Compatibility of competing biochemical modules: capacity, recovery, and finite operation

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Abstract

When two biochemical modules draw on one replenished resource and keep private chemical states, "compatible" is not one property. We separate three questions and answer each for an explicit eight-state model in which a glutathione peroxidase branch and a peroxiredoxin–thioredoxin branch share NADPH. *Capacity*: stationary branch responses decide whether a joint operating point exists; this sharp criterion is inherited from a companion paper. *Recovery*: we prove that every state of an explicit preparation region has a unique physical solution that meets both service quotas permanently after 4 ms, with exact bounds on shortfall and regeneration expenditure; this statement is verified in Lean 4. An exact-arithmetic tube certificate extends it to every preparation within 0.1% of a service-deficient state in each concentration independently. In contrast, a one-parameter family of repair kinetics has identical stationary states at every source strength but a necessary recovery delay growing like $1/\sigma$, so no deadline can be read from stationary data. *Duration*: an exact storage account shows that a finite regeneration donor cannot support positive service forever, and that finite missions are decided by the actually depleted trajectory. For a one-second mission under a linear donor law, a conservative fixed-band argument needs a 1.9 M stock; a moving-tube certificate proves that 120 μM suffices and that 60 μM fails, from the same preparations. During the successful mission the source falls below the stationary compatibility threshold for more than half the time, while both quotas remain met: stored carrier, not instantaneous capacity, pays the difference. Stationary thresholds therefore can both underestimate readiness time and overestimate finite-mission supply. All conclusions concern the declared maintained model; they assert no global attraction, biological calibration, or clinical meaning.

1 Introduction

Modules that are adequate alone can fail together. The best-studied mechanisms are loading of an upstream module by its downstream client, known as retroactivity [11], and competition for a limited pool of shared machinery, which reshapes the responses of genetic circuits and can be designed around [13, 20, 25]. Redox metabolism offers a chemically explicit instance: glutathione peroxidase and the peroxiredoxin–thioredoxin system both remove hydrogen peroxide, and both are restored by NADPH through their own reductases [16, 21, 26]. The kinetic model of Adimora et al. [1] already couples these branches. None of these observations is new here.

What is usually left implicit is what a claim of compatibility *means*. A stationary calculation answers whether some joint operating point exists. It does not say whether a given preparation reaches it, when, or for how long a finite supply can hold it. This paper states the question with all of its quantifiers and answers it for one declared model.

> *Given two modules that share a replenished resource and retain private chemical states, what must be known to guarantee both services from a specified preparation, by a deadline, for a stated duration?*

Definition 1.1 (Operating specification)**.** Fix a model, a preparation set $\mathcal{P}_0$ of initial states, a supply law, service outputs H_i with quotas $q_i > 0$, a deadline $\tau \geq 0$, a horizon $\tau \leq H \leq \infty$, and shortfall allowances $d_i \geq 0$. The specification holds if every $u_0 \in \mathcal{P}_0$ has a unique solution on $[0, H]$ that remains physical and satisfies

$$H_i(u(t)) \geq q_i \quad (\tau \leq t \leq H), \qquad \int_0^H [q_i - H_i(u(t))]_+ \,\mathrm{d}t \leq d_i. \tag{1}$$

Resource use is accounted on that same solution. $H = \infty$ is admissible only when the supply law permits indefinite operation.

The definition only fixes quantifiers. The content of the paper is three results about it, which are complementary and not a chain of equivalences.

(i) **Capacity.** Stationary branch responses and the shared source decide whether a joint operating point exists, and give the attained minimum source strength $s_{\min}$. This is the sharp criterion of the companion paper [7]; we restate it in Section 3 and use it only as a benchmark.

(ii) **Recovery.** Retained private states and kinetic clocks decide whether a preparation becomes productive by a deadline. We give a constructive certificate (Theorems 4.2 and 4.4, Corollary 4.3) and an obstruction: a matched family of repair kinetics with identical stationary states and unbounded necessary delay (Propositions 3.2 and 3.4).

(iii) **Duration.** Stored carrier can pay for service that the instantaneous source cannot, but a finite donor cannot do so forever (Proposition 2.1). For a finite mission the relevant object is the depleted trajectory (Theorems 5.1 and 5.3, Corollary 5.2).

What is inherited and what is new. Table 1 separates the two. The stationary theory, including private-state elimination, monotone branch currents, the attained threshold, uniqueness of the stationary state, kinetic overdelivery and enzyme redesign, belongs to [7] and is not reproved. Everything dynamic is new: an all-initial-state recovery theorem with its accounts, its extension to an independent 0.1% preparation box, source robustness, three finite-donor results, and the hidden-clock obstruction. Two related companion papers treat different questions and are not used: permanence of arbitrarily many scalar consumers on a dynamic reservoir [8] concerns persistence of abundance, not service quotas of retained-state modules; reliable operation of coupled autocatalytic reactors [9] gives stochastic finite-count guarantees, which we do not import into this deterministic setting.

Relation to set-invariance and validated-integration methods. The certificates are polyhedral barrier arguments in the tradition of Nagumo [17]; see Blanchini [5] for invariance in control and Prajna and Jadbabaie [19] for barrier certificates. Polyhedral norms also underlie the structural nonexpansivity and contraction theory of Al-Radhawi et al. [2]. Our certificates are specific to one rational vector field on a bounded region; they prove finite-time capture and service, and they do not prove contraction, convergence to an equilibrium, or robustness to arbitrary rates. A contraction claim would need a bound on the derivative of the nonlinear remainder, not only on its size. The kinetics also admit no orthant order that makes them cooperative [7], so monotone-systems tools [3] do not apply directly. Validated integrators such as Flow* [6] and CAPD [15], and δ-decision procedures [12], are natural tools for discovering finite-horizon enclosures, and rigorous and formally verified ODE enclosures have a substantial history [14, 23]. We make no claim of outperforming them and no claim of priority in formalizing ODE arguments. The contribution is an operational theorem with an infinite-horizon tail that is kernel-checked, and finite-horizon certificates that reduce to a list of rational inequalities anyone can replay.

Table 1: Inherited and new results. "Lean" means a kernel-checked statement; "exact" means a replayable certificate in exact rational arithmetic that is not kernel-checked; "conventional" means a complete proof in this paper.

Result	Status	Evidence
Stationary criterion, $s_{\min}$, uniqueness, overdelivery (Thm. 3.1)	inherited [7]	Lean (there)
Recovery in 4 ms from all of $\mathcal{R}$, shortfall, expenditure (Thm. 4.2)	new	Lean
Recovery from an independent 0.1% box (Cor. 4.3)	new	exact + Lean
Continuous source uncertainty (Thm. 4.4)	new	conventional; margins Lean
Stationary equivalence and repair clock (Props. 3.2, 3.4)	new	Lean lemmas; conventional
Storage account, no perpetual service (Prop. 2.1)	new	conventional
Linear donor, fixed band (Thm. 5.1)	new	conventional; margins Lean
Saturating donor (Cor. 5.2)	new	conventional; algebra Lean
Linear donor at 120 and 60 μM (Thm. 5.3)	new	exact

2 Model, services, and accounts

2.1 The declared model

Time is in seconds and concentrations in μM. The state, with coordinates numbered 0 to 7, is

$$u = (x, z, e_1, e_2, z_T, h, w, v),$$

where x is reduced NADPH. The GPx-side private states are z, e_1, e_2, and the Trx/Prx-side private states are z_T, h, w, v; the variable w is the hyperoxidized peroxiredoxin population, which returns only through slow repair [4, 27]. Four pools are reconstructed by conservation:

$$g = 371.56 - 2z - e_2, \quad e_0 = 50 - e_1 - e_2, \quad y = .505 - z_T, \quad r = 19.096 - h - w - v. \tag{2}$$

The service and internal currents are

$$\begin{aligned} H_G &= .21e_0, & H_T &= 2.1yv, \\ \phi_1 &= .04ge_1, & \phi_2 &= 10ge_2, \\ \psi_G &= 3.2x(z - 1.78), & \psi_T &= 20x(z_T - .075), \\ \theta_r &= .4r, & \theta_w &= .00072\,\sigma h, \\ \theta_h &= .003\,\sigma w, & \theta_v &= 15h, \end{aligned} \tag{3}$$

and the shared regeneration current is

$$R_s(x) = 375\,s\,\frac{30 - x}{87.3 - x}. \tag{4}$$

The vector field $F_{s,\sigma}$ is

$$\begin{aligned} \dot{x} &= R_s(x) - \psi_G - \psi_T, & \dot{z} &= \phi_2 - \psi_G, \\ \dot{e}_1 &= H_G - \phi_1, & \dot{e}_2 &= \phi_1 - \phi_2, \\ \dot{z}_T &= H_T - \psi_T, & \dot{h} &= \theta_r + \theta_h - \theta_w - \theta_v, \\ \dot{w} &= \theta_w - \theta_h, & \dot{v} &= \theta_v - H_T. \end{aligned} \tag{5}$$

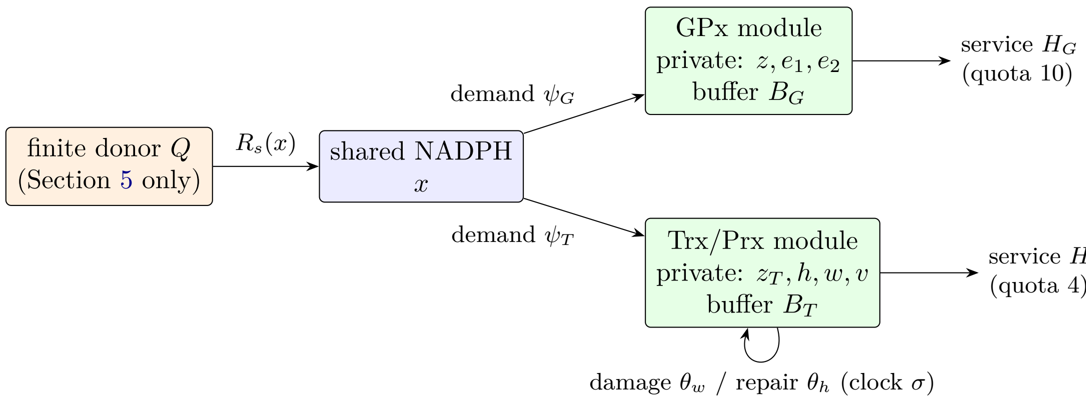

Figure 1: Accounts in the shared-NADPH model. Service H_i and resource demand ψ_i differ whenever a module's buffer B_i changes. Arrows summarize currents of the declared ODE, not elementary stoichiometry. Peroxide and the repair donor are maintained; the regeneration donor is finite only in Section 5.

The scales s and σ are dimensionless; nominal operation is $s = 3/25$, $\sigma = 1$. Every terminating decimal is an exact rational number. The *physical region* is

$$0 \le x \le 30, \quad z \ge 1.78, \quad z_T \ge .075, \quad g, e_0, e_1, e_2, y, r, h, w, v \ge 0, \tag{6}$$

on which $87.3 - x \ge 57.3$, so the source has no pole. The service quotas are $q_G = 10$ and $q_T = 4\ \mu\mathrm{M\,s}^{-1}$, as in [7].

Maintained inputs and provenance. The model is the maintained-peroxide subnetwork of [7], motivated by Adimora et al. [1] and Pannala and Dash [18]. The rates in (3) include a maintained peroxide challenge. Repair of w is phenomenological and its energy donor is not a state. The coefficients are a declared reduced model; they are not attributed individually to [1] and are not fitted here. Quotas, preparations and certificate radii are mathematical design choices. The model is not a thermochemically closed cell, and the quotas are not health thresholds.

2.2 Service, demand, and storage

The service currents H_G, H_T are not the instantaneous NADPH demands ψ_G, ψ_T. Let

$$B_G = z - 1.78 + e_1 + e_2, \qquad B_T = z_T - .075, \qquad W = 234.43 + x - B_G - B_T. \tag{7}$$

Direct differentiation of (5) gives, for every s and σ,

$$\dot{B}_G = H_G - \psi_G, \qquad \dot{B}_T = H_T - \psi_T, \qquad \dot{W} = R_s - H_G - H_T. \tag{8}$$

On the physical region $B_G \le 234$ (from $2z + e_2 \le 371.56$ and $e_1 + e_2 \le 50$) and $B_T \le .43$, so $W \ge 0$. The constant 234.43 only normalizes W to be nonnegative; differences of W, not its origin, enter every bound. The account (8) is for regeneration against service. It must not be merged with peroxide removal: since $\dot{r} = H_T - \theta_r$,

$$H_G + \theta_r + \theta_w = H_G + H_T + \theta_h + \dot{w} - \dot{r}, \tag{9}$$

so peroxide consumption, service, and repair differ in transients.

The storage identity is not specific to this chemistry.

Proposition 2.1 (Storage account)**.** *Let modules $i = 1, \dots, m$ have buffers with $\dot{B}_i = H_i - D_i$, let the shared resource satisfy $\dot{x} = J - \sum_i D_i$, and let a donor satisfy $\dot{Q} = -J$, all in common stoichiometric equivalents. Put $W = x - \sum_i B_i + C$ for a constant C. Then*

$$(Q + W)' = -\sum_i H_i. \tag{10}$$

If $W \geq 0$ and $Q \geq 0$ along a solution on $[a, b]$, then $\int_a^b \sum_i H_i \, \mathrm{d}t \leq Q(a) + W(a)$. In particular, total service $\sum_i H_i \geq q > 0$ on $[a, b]$ forces

$$b - a \leq \frac{Q(a) + W(a)}{q},$$

and no solution with finite initial donor, no replenishment, $Q \geq 0$ and $W \geq 0$ has $\sum_i H_i \geq q > 0$ for all large times. [conventional]

Proof. Add the three balance laws to get (10); integrate and use $Q(b) + W(b) \geq 0$. □

The proposition is a conservation statement under its stated units. It is not a thermodynamic theorem, and it gives no useful lower bound on the stock needed for a short mission, because $W(a)$ may be large. That question is settled by trajectories in Section 5.

3 Capacity, and what it cannot determine

3.1 The inherited stationary criterion

Eliminating private states at stationarity gives each branch a strictly increasing service current $j_i(x)$, so that a quota $j_i \geq q_i$ is an NADPH floor $x \geq X_i$. Write $J = j_G + j_T$ and $L = \max_i X_i$.

Theorem 3.1 ([7])**.** *For $\sigma = 1$ and quotas $(10, 4)$, a physical stationary state of $F_{s,1}$ meeting both quotas exists if and only if*

$$s \geq s_{\min} = \frac{J(L)}{R_1(L)}, \qquad L = X_T = \frac{460180}{489387}, \qquad .11371266 \leq s_{\min} \leq .11371268. \tag{11}$$

For each s the full stationary state is unique. The numerator is kinetic demand $J(L) \approx 14.3489$, which exceeds the quota sum 14 by the kinetic overdelivery of the GPx branch. [inherited; Lean in the companion paper]

The nominal source $s = .12$ exceeds $s_{\min}$ by 5.5%, and the setting $s = .1$ fails. We use (11) only as the stationary benchmark. A redesign of an enzyme inventory, as in [7], changes the transient field, so it does not inherit the dynamic certificates below without a new check.

3.2 A matched family with the same stationary states

Proposition 3.2 (Matched repair rates)**.** *For every s and every $\sigma > 0$, the stationary states of $F_{s,\sigma}$ are exactly those of $F_{s,1}$. Hence every stationary service curve, quota floor, and threshold, in particular Theorem 3.1, is the same for all $\sigma > 0$.* [Lean]

Proof. At stationarity the w equation reads $\sigma(.00072h - .003w) = 0$, which is equivalent to its $\sigma = 1$ version. When it holds, the two exchange terms cancel in the h equation. The other six equations do not contain σ. Both implications hold. □

This is equality of stationary equations. It is not a time reparametrization of the transient system, because only the two exchange currents θ_w, θ_h are rescaled.

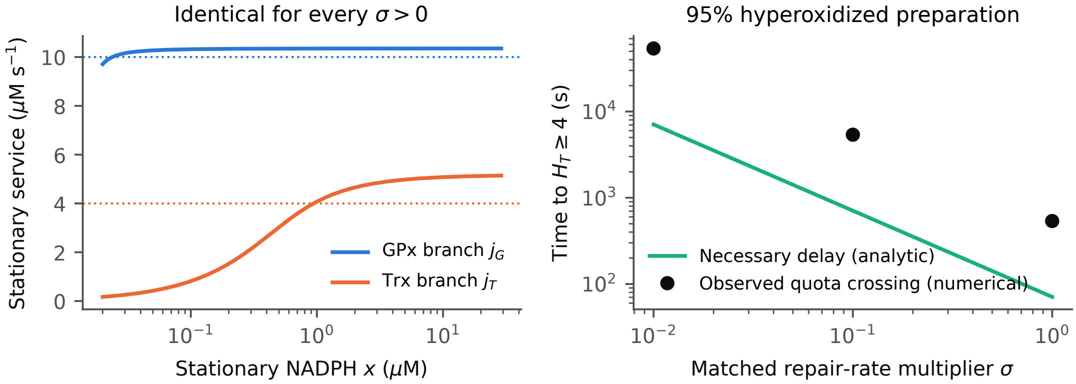

Figure 2: Stationary information loses the repair clock. Left: the eliminated stationary service responses, with quotas dotted, are identical for all $\sigma > 0$. Right: the necessary delay (13) for the 95% hyperoxidized preparation, and numerical crossing times, which are observations and not certified deadlines.

3.3 The hidden clock

Proposition 3.3 (Hidden clock, general form)**.** *Let $w : [0, t_1] \to [0, \infty)$ be C^1 with $\dot{w} \geq -\kappa\sigma w$ for constants $\kappa, \sigma > 0$, and suppose a service condition implies $w \leq w_q$ for some $0 < w_q < w(0)$. Then every time t at which the service condition holds satisfies*

$$t \geq \frac{1}{\kappa\sigma} \log \frac{w(0)}{w_q}. \tag{12}$$

Consequently, if a set of descriptors is the same for all $\sigma > 0$, no finite deadline that is a function of those descriptors and of $w(0)$ alone is valid for all $\sigma > 0$. [conventional]

Proof. $(e^{\kappa\sigma t} w)' \geq 0$, so $w(t) \geq w(0)e^{-\kappa\sigma t}$; service at time t needs $w(0)e^{-\kappa\sigma t} \leq w_q$. The right side of (12) is unbounded as $\sigma \to 0$. □

Proposition 3.4 (Repair clock in the model)**.** *Let u be a physical solution of (5) with any source law and $\sigma > 0$. Put $w_q = 19.096 - 4/(2.1 \cdot .43) \approx 14.666$. If $w(0) > w_q$, every t with $H_T(u(t)) \geq 4$ satisfies*

$$t \geq \frac{\log\big(w(0)/w_q\big)}{.003\,\sigma}. \tag{13}$$

[conventional; the inequality $\dot{w} \geq -.003\sigma w$ and its integrating-factor form are Lean lemmas]

Proof. Since $h \geq 0$, $\dot{w} = \sigma(.00072h - .003w) \geq -.003\sigma w$. Physicality gives $y \leq .43$ and $v \leq 19.096 - w$, so $H_T \leq 2.1(.43)(19.096 - w)$ and $H_T \geq 4$ implies $w \leq w_q$. Apply Proposition 3.3. □

For the physical preparation with $w(0) = .95 \cdot 19.096$ the bound is $70.877/\sigma$ seconds. It holds at every regeneration strength. It is an $\Omega(1/\sigma)$ *necessary* delay: it is not an upper bound, not a sharp asymptotic, and it does not prove eventual recovery. Numerical quota crossings for this preparation are later still, near 537 s at $\sigma = 1$ (Figure 2). What the two propositions establish together is an exact non-identifiability: stationary states, stationary service curves and pool totals do not determine the repair clock, so they determine no deadline. A modeler needs a bound on initial retained damage and the repair kinetics. Whether those can be estimated from a given assay is a separate, practical identifiability question; a derivative-based estimate of σ is ill-conditioned near repair balance.

4 A constructive operating certificate

4.1 A moving-box lemma

The positive results rest on one elementary lemma about time-dependent boxes. It is a finite simultaneous-face version of Nagumo's tangency condition [5, 17], stated so that corners, boundary initial states and piecewise-affine motion need no separate treatment.

Lemma 4.1 (Moving box)**.** *Let $a : [t_0, t_1] \to \mathbb{R}^n$ be a C^1 solution of $\dot{a} = f(t, a)$ and let $q : [t_0, t_1] \to [0, \infty)^n$ be C^1. Suppose that for every $t \in [t_0, t_1]$, every index i, every sign $\eta \in \{-1, 1\}$, and every a with $|a_j| \le q_j(t)$ for all j and $a_i = \eta q_i(t)$,*

$$\eta f_i(t, a) < \dot{q}_i(t). \tag{14}$$

If $|a_j(t_0)| \le q_j(t_0)$ for all j, then $|a_j(t)| \le q_j(t)$ for all j and all $t \in [t_0, t_1]$. [conventional; the fixed-rate version used for Theorem 4.2 is a Lean lemma]

Proof. Let t^* be the supremum of times up to which all $2n$ inequalities hold. By continuity they hold at t^*. Suppose $t^* < t_1$. For a face that is active at t^*, the function $g = q_i - \eta a_i$ has $g(t^*) = 0$ and, by (14) applied at the point $a(t^*)$, $\dot{g}(t^*) > 0$; so $g > 0$ on an interval to the right of t^*. For an inactive face $g(t^*) > 0$ and continuity gives the same. There are finitely many faces, so all inequalities hold slightly beyond t^*, a contradiction. □

The lemma is applied on consecutive time intervals and joined at the endpoints, where only the inclusion is needed. With a moving center $c(t)$ one applies it to $a = M^{-1}(u - c(t))$, whose field contains the drift term $-M^{-1}\dot{c}(t)$; omitting that term would invalidate the argument.

4.2 Geometry and the nominal theorem

The certificate fixes an invertible rational matrix M, its exact inverse, and a rational center

$$c \simeq (1.661997,\ 3.726108,\ .710658,\ .00284263,\ .211599,\ .302703,\ .0726486,\ 7.369302). \tag{15}$$

$M = \widehat{Q}\,\mathrm{diag}(\rho)$, where $\widehat{Q}$ is a ten-digit rational rounding of an eigenvector matrix of $DF_{3/25,1}$ near its equilibrium and ρ is a vector of scales. M approximately separates linear modes, with rates from about 3641 s^{-1} (the intermediate e_2) to .003 s^{-1} (repair); it is not an exact diagonalization, c is not assumed stationary, and every residual coupling is bounded in the proof. The exact data are in the supplementary file `data/modal_box.json` (Appendix B). For $q \in \mathbb{R}^8_{\ge 0}$ put $\mathcal{B}(q) = \{c + Ma : |a_i| \le q_i\}$ and

$$\begin{aligned} &R = (.9, .834, .7, .7, .7, .7, .7, .7), && S = (.9, .7, .7, .7, .7, .7, .7, .7), \\ &\mathcal{R} = \mathcal{B}(R), \quad \mathcal{S} = \mathcal{B}(S), && \tau = 1/250. \end{aligned} \tag{16}$$

These are correlated parallelepipeds. A state u_0 lies in $\mathcal{R}$ if and only if $|M^{-1}(u_0 - c)| \le R$ componentwise. A reference deficient preparation is

$$\begin{aligned} p = c + \tfrac{5}{6} M_{\cdot 1} \simeq (&1.692492,\ 3.730262,\ .710652,\ .00284261, \\ &.249285,\ .302574,\ .0726486,\ 7.357791), \end{aligned} \tag{17}$$

which has $H_T(p) \approx 3.951 < 4$, and $\mathcal{P} = \{u : |u_j - p_j| \le 5 \cdot 10^{-7}\}$ is an independent box around it.

Theorem 4.2 (Nominal operational service)**.** *Every state in $\mathcal{P}$ is physical, lies in $\mathcal{R}$, and has $H_T < 4$. For every $u_0 \in \mathcal{R}$, the system (5) with $(s, \sigma) = (3/25, 1)$ has a unique forward* *solution u with $u(0) = u_0$. It is physical and lies in $\mathcal{R}$ for all $t \geq 0$, and lies in $\mathcal{S}$ for all $t \geq \tau$. On this solution*

$$H_G \geq \tfrac{207}{20}\ (t \geq 0), \qquad H_T \geq \tfrac{391}{100}\ (t \geq 0), \qquad H_T \geq \tfrac{401}{100}\ (t \geq \tau). \tag{18}$$

For every $T \geq 0$,

$$\int_0^T [10 - H_G]_+ \,\mathrm{dt} = 0, \qquad \int_0^T [4 - H_T]_+ \,\mathrm{dt} \leq \frac{9}{25000}, \tag{19}$$

and for every $a \geq \tau$ and $T \geq 0$,

$$\int_a^{a+T} H_G \,\mathrm{dt} \geq 10.35\,T, \quad \int_a^{a+T} H_T \,\mathrm{dt} \geq 4.01\,T, \quad \int_a^{a+T} R_{3/25}(x) \,\mathrm{dt} \geq 14.36\,T - .1. \tag{20}$$

Thus Definition 1.1 holds with $\mathcal{P}_0 = \mathcal{R}$, $\tau = 4$ ms, $H = \infty$, $d_G = 0$ and $d_T = 3.6 \cdot 10^{-4}$ μM. [Lean]

Existence, physicality, capture and uniqueness are conclusions. The theorem does not assume an attracting equilibrium or that a proposed trajectory stays physical. An earlier certificate for the same field had deadline 2000 s, because it shrank every modal coordinate at a common slow rate. The factor $5 \cdot 10^5$ between the two deadlines measures certificate conservatism, not chemistry: service recovery needs only the second-fastest mode to relax, while the slow repair coordinate may stay where it is.

4.3 Proof of Theorem 4.2

Field expansion. Write $a = M^{-1}(u - c)$ and $f(a) = M^{-1}F_{3/25,1}(c + Ma)$, and let $A = M^{-1}DF_{3/25,1}(c)M$ and $b = M^{-1}F_{3/25,1}(c)$. Then

$$f(a) = b + Aa + N(a), \qquad |N_i(a)| \leq n_i \quad (|a_j| \leq 1), \tag{21}$$

with rational n_i obtained as follows. Put $d = Ma$ and $w_j = \sum_k |M_{jk}|$. The polynomial part of F is quadratic, and its second-order remainder is a fixed linear combination of five products,

$$d_0 d_1, \quad d_0 d_4, \quad (-2d_1 - d_3)d_2, \quad (-2d_1 - d_3)d_3, \quad d_4 d_7,$$

bounded by $w_0 w_1$, $w_0 w_4$, $(2w_1 + w_3)w_2$, $(2w_1 + w_3)w_3$, $w_4 w_7$. The source remainder is exactly

$$-\frac{45\,(57.3)\,d_0^2}{(87.3 - c_0)^2(87.3 - c_0 - d_0)}, \tag{22}$$

bounded using $87.3 - c_0 - w_0 > 0$. The reaction vectors are transformed by M^{-1} *before* absolute values are taken; this keeps cancellations that a coordinatewise comparison of the raw field loses, and is why the modal certificate succeeds where absolute comparisons in physical coordinates fail. The same enclosures show that the whole unit cube $\mathcal{B}(\mathbf{1})$ is strictly physical, with $1.60 < x < 1.73$.

Face budgets. For $q \leq \mathbf{1}$ define

$$D_i(q) = A_{ii} q_i + \sum_{j \neq i} |A_{ij}| q_j + |b_i| + n_i. \tag{23}$$

On a face $a_i = \eta q_i$ with $|a_j| \leq q_j$, (21) gives $\eta f_i(a) \leq D_i(q)$. Let $v = (0, -67/2, 0, \ldots, 0)$. Exact evaluation (Table 2) gives

$$D_i(R) < v_i, \qquad D_i(S) < v_i \qquad (0 \leq i < 8). \tag{24}$$

Table 2: Signed face budgets (rounded; the strict inequalities use exact rationals). Only modal coordinate 1 has a moving face.

i	$D_i(R)$	$D_i(S)$	v_i	i	$D_i(R)$	$D_i(S)$	v_i
0	-3274.3640	-3274.3652	0	4	$-.33539200$	$-.33539200$	0
1	-41.1928	-34.3435	-33.5	5	$-.11454379$	$-.11454379$	0
2	$-.66394145$	$-.66394147$	0	6	$-.02972857$	$-.02972857$	0
3	$-.77105616$	$-.77105622$	0	7	$-.00049735$	$-.00049742$	0

Capture, existence, physicality. On $[0, \tau]$ let $q(t) = (1 - 250t)R + 250tS$, so $\dot{q} = v$. Since D_i is affine in q, (24) gives $D_i(q(t)) < v_i$, which is (14). Lemma 4.1 keeps the solution in $\mathcal{B}(q(t))$, including from boundary and corner initial states. At $t = \tau$ the solution is in $\mathcal{S}$; since $D_i(S) < v_i \leq 0$, the lemma with constant radius makes $\mathcal{S}$ forward invariant, and likewise $\mathcal{R}$. The field is smooth on $x < 87.3$ and the barriers confine any local solution to a compact subset of that domain, so the solution continues for all time and is unique. The Lean development realizes the same continuation through a globally Lipschitz extension of the field that agrees with it on the invariant region.

Outputs, preparation, accounts. For a radius q let $d_j(q) = \sum_k |M_{jk}|q_k$. From $|u_j - c_j| \leq d_j(q)$,

$$H_G \geq L_G(q) = .21(50 - c_2 - c_3 - d_2 - d_3), \qquad H_T \geq L_T(q) = 2.1(.505 - c_4 - d_4)(c_7 - d_7), \tag{25}$$

with $L_G(R), L_G(S) > 10.35$, $L_T(R) > 3.91$, $L_T(S) > 4.01$. For $u \in \mathcal{P}$, $|[M^{-1}(u-c)]_i| \leq \frac{5}{6}\mathbf{1}_{i=1} + \varepsilon \sum_j |(M^{-1})_{ij}| \leq R_i$ with $\varepsilon = 5 \cdot 10^{-7}$, and $H_T(u) \leq 2.1(.505 - p_4 + \varepsilon)(p_7 + \varepsilon) < 3.9512$. The floors give (19): the Trx shortfall is at most .09 for a time τ. With $\ell = (1, -1, -1, -1, -1, 0, 0, 0)$ one has $W(c + Ma) = W(c) + \ell Ma$, and on $\mathcal{S}$ the range of W is

$$2 \sum_i |(\ell M)_i| S_i = \frac{2430664273305669}{25000000000000000} < \frac{1}{10}. \tag{26}$$

Integrating (8) along the solution, $\int_a^{a+T} R_{3/25}\,\mathrm{d}t = \int_a^{a+T} (H_G + H_T)\,\mathrm{d}t + W(u(a+T)) - W(u(a))$, which gives the last bound of (20). The subtracted .1 is storage on $\mathcal{S}$; it does not come from summing quotas. □

4.4 Preparation geometry and an independent tolerance

The box $\mathcal{P}$ has halfwidth 0.5 pM, which is a legitimate concern if read as the tolerance of the theorem. It is not: the theorem covers all of $\mathcal{R}$, whose coordinate projections have halfwidths

$$(48,\ 27,\ .15,\ .0012,\ 39,\ 1.4,\ .11,\ 53)\ \text{nM}$$

in the order of u. The projections are wide but correlated, and their product is not contained in $\mathcal{R}$. For an independent box with center p' and halfwidths ϵ_j, a sufficient membership test is the finite linear condition

$$|M^{-1}(p' - c)| + |M^{-1}|\,\epsilon \leq R. \tag{27}$$

Around p, the largest uniform halfwidth allowed by (27) is 0.633 pM. The binding row is modal coordinate 0, the 3641 s^{-1} mode of the intermediate e_2, whose scale $\rho_0 = 10^{-6}$ is tiny because $e_2 \approx 2.8$ nM is itself tiny. Coordinate by coordinate the test allows $x \pm 0.48$ nM, $z \pm 2.0$ nM, $v \pm 2.6$ nM, $w \pm 0.11$ nM, but only $e_2 \pm 0.64$ pM. The narrow box is therefore a property of the chosen modal scales in rapidly equilibrating directions, not of the chemistry, and it can be removed without changing the Lean theorem: fast directions may start far outside $\mathcal{R}$ and still arrive in $\mathcal{S}$ quickly.

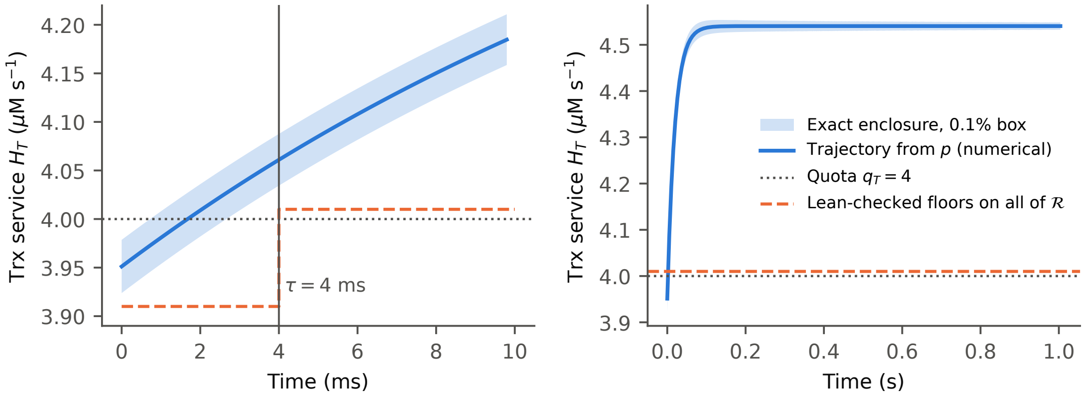

Figure 3: Recovery under the maintained source $s = .12$. The band is the exact-arithmetic enclosure of H_T over all preparations in the 0.1% box $\mathcal{P}_\kappa$ (Corollary 4.3); the solid curve is a numerical solution from p. Dashed: the Lean-checked floors of Theorem 4.2, valid on all of $\mathcal{R}$. The numerical curve crosses the quota near 1.68 ms; the curve is an illustration, the band and floors are proofs.

Corollary 4.3 (Independent 0.1% preparations)**.** *Let $\mathcal{P}_\kappa = \{u : |u_j - p_j| \le \kappa p_j,\ 0 \le j < 8\}$ with $\kappa = 10^{-3}$. Then $\mathcal{P} \subset \mathcal{P}_\kappa$, every state of $\mathcal{P}_\kappa$ is physical and has $H_T < 4$, and for every $u_0 \in \mathcal{P}_\kappa$ the nominal system has a unique global physical solution with*

$$H_G \ge 10\ (t \ge 0), \qquad H_T \ge 4\ (t \ge \tau), \qquad \int_0^\infty [4 - H_T]_+\, \mathrm{d}t \le 1.1 \cdot 10^{-4}\ \mu\mathrm{M},$$

and $u(t) \in \mathcal{S}$ for all $t \ge 1/4 + \tau$, after which all conclusions of Theorem 4.2 hold. [exact tube on $[0, 1.004]$, then Lean]

Proof. $\mathcal{P}_\kappa \subset p + M\mathcal{B}_0(q_0)$, where $\mathcal{B}_0(q) = \{a : |a_j| \le q_j\}$ and

$$q_{0,i} = \kappa \sum_j |(M^{-1})_{ij}|\, p_j, \qquad q_0 \approx (8.06,\ .0101,\ 10.7,\ 27.0,\ 2.39,\ .207,\ .303,\ .483);$$

coordinates $0, 2, 3, 4$ start far outside $\mathcal{R}$. The tube certificate of Appendix A with the maintained source verifies the hypotheses of Lemma 4.1 on 1206 steps covering $[0, 1.004]$, with service floors $H_G \ge 10.3499$ throughout and $H_T \ge 4$ from 2.75 ms, and $H_T \ge 4.034$ on $[\tau, 1.004]$. The slice at the grid time $t^* = 6191921/25000000 < 1/4$ satisfies $|M^{-1}(\bar{c}(t^*) - c)| + q(t^*) \le S$ componentwise, so $u(t^*) \in \mathcal{S} \subset \mathcal{R}$. Theorem 4.2 then applies from t^*: $u(t) \in \mathcal{S}$ for $t \ge t^* + \tau$, while the tube covers $[0, t^* + \tau]$. The shortfall bound is the exact sum $\sum_k h_k [4 - L_{T,k}]_+$ over steps before τ. □

The independent tolerances are now $x \pm 1.7$ nM, $z \pm 3.7$ nM, $e_1 \pm 0.71$ nM, $e_2 \pm 2.8$ pM, $z_T \pm 0.25$ nM, $h \pm 0.30$ nM, $w \pm 73$ pM and $v \pm 7.4$ nM, simultaneously. The binding quantity is no longer geometry but the service margin: a 0.2% box still passes the mission checks in floating point but leaves $H_T - 4 < .011$ at τ and does not enter $\mathcal{S}$ by one second, so we do not claim it. These are deterministic neighborhoods. They are not stochastic confidence statements; intrinsic-noise claims would need a compartment volume and a stochastic model [cf. 9].

4.5 Continuous source uncertainty

Theorem 4.4. *Let $s : [0, \infty) \to \mathbb{R}$ be continuous with $|s(t) - 3/25| \le \delta = 10^{-6}$. For every $u_0 \in \mathcal{R}$ the system $\dot{u} = F_{s(t),1}(u)$ has a unique global forward solution, and all conclusions* *of Theorem 4.2 hold, with the expenditure bound stated for the actual current* $R_{s(t)}(x(t))$. [conventional; inequality (28) Lean]

Proof. $M^{-1}(F_{s,1}(u) - F_{3/25,1}(u)) = M^{-1}e_x\,(s - 3/25)R_1(x)$ and $0 \le R_1 \le 130$ on the physical region, so the modal perturbation in coordinate i is at most $E_i = 130\,\delta\,|(M^{-1})_{i0}|$. Exact checks give

$$D_i(R) + E_i < v_i, \qquad D_i(S) + E_i < v_i. \tag{28}$$

The perturbed face condition (14) follows by affine interpolation, pointwise in s and with no derivative of s. The field is continuous in t and locally Lipschitz in u, so Picard–Lindelöf, Lemma 4.1 and compact continuation [24] give the solution. Output bounds depend only on the rectangles, and (8) holds with the actual source. □

The band is a sufficient certificate, not an uncertainty model or a robustness threshold; discontinuous switching is outside the statement.

5 Costs and finite duration

Append a donor stock Q, in donor equivalents per reaction volume, consumed one equivalent per NADPH regenerated. Peroxide and the repair donor remain maintained. By Proposition 2.1, for any such law

$$(Q + W)' = -H_G - H_T, \tag{29}$$

so no finite stock supports the quotas forever. What a finite stock can do depends on how the source responds to depletion.

5.1 Linear donor law: a fixed-band argument

For a concentration scale $V > 0$ let

$$\dot u = F_{Q/V,1}(u), \qquad \dot Q = -R_{Q/V}(x), \qquad Q(0) = Q_0 = \tfrac{3}{25}V. \tag{30}$$

V converts stock to the source scale; it is not a geometric volume.

Theorem 5.1 (Finite mission, fixed band). *Let $H \ge \tau$ and $V \ge 16{,}000{,}000\,H$. For each $u_0 \in \mathcal{R}$, (30) has a unique solution on $[0, H]$; its u component is physical, stays in $\mathcal{R}$, lies in $\mathcal{S}$ on $[\tau, H]$, and satisfies the floors and shortfall bounds of Theorem 4.2. Moreover*

$$Q_0 - 16t \le Q(t) \le Q_0, \qquad 14.36\,T - .1 \le Q(a) - Q(a+T) \le 16\,T \quad (\tau \le a \le a+T \le H). \tag{31}$$

[conventional nine-state proof; margins (28) Lean]

Proof. The nine-state field is smooth for $x < 87.3$. Take the sixteen modal faces together with $Q \le Q_0$ and $Q \ge Q_0 - 16t$. On this prospective tube $|Q/V - 3/25| \le 16H/V \le \delta$ and $Q \ge (3/25 - \delta)V > 0$, so (28) applies on the tube without assuming that the solution stays in it. On the upper donor face $(Q_0 - Q)' = R > 0$ because $x < 30$. On the lower one $(Q - Q_0 + 16t)' = 16 - R_{Q/V}(x) > 0$, since $(30 - x)/(87.3 - x)$ is decreasing and $R_{Q/V}(x) \le (3/25 + \delta)\,375 \cdot 30/87.3 = 120001/7760 < 16$. Lemma 4.1, applied to all eighteen faces, and compact continuation give the solution. The bounds (31) follow from (29), the floors, and (26). □

For the one-second service window $[.004, 1.004]$ this needs $V = 16{,}064{,}000$, that is $Q_0 = 1{,}927{,}680\ \mu$M ≈ 1.9 M, to spend at most $16.1\ \mu$M. The reason is structural. The proof keeps the source inside the band of Theorem 4.4, and under the linear law that allows spending only

$$\frac{Q_0 - Q}{Q_0} \le \frac{10^{-6}}{.12} \approx 8.3 \text{ parts per million} \tag{32}$$

of the stock. Tightening the bound 16 on the regeneration rate cannot repair a five-order ratio. Three quantities must be kept apart: equivalents consumed; inventory the *kinetic law* needs to hold its rate; and reserve added by the *certificate*. The 1.9 M figure mixes the last two. We remove the second by changing the law (Section 5.2) and the third by changing the proof (Section 5.3).

5.2 A saturating donor law

Let the source depend on the donor through a saturating factor with affinity scale $K > 0$, normalized at the initial stock:

$$s_{\mathrm{eff}}(Q) = s_0 \frac{Q/(K+Q)}{Q_0/(K+Q_0)}, \qquad \dot{u} = F_{s_{\mathrm{eff}}(Q),1}(u), \qquad \dot{Q} = -R_{s_{\mathrm{eff}}(Q)}(x), \tag{33}$$

with $s_0 = 3/25$ and $Q(0) = Q_0$. This is a different model from (30), not a sharper analysis of it. One equivalent is still consumed per equivalent regenerated, so (29) holds.

Corollary 5.2 (Saturating donor)**.** *Let $H \geq \tau$, $D = 16H$, $\delta = 10^{-6}$, and $Q_0 > D$ with*

$$\frac{s_0 K D}{Q_0\,(K + Q_0 - D)} \leq \delta, \quad \textit{equivalently} \quad Q_0 \geq \frac{D - K + \sqrt{(D-K)^2 + 4(s_0/\delta)KD}}{2}. \tag{34}$$

Then every conclusion of Theorem 5.1 holds for (33) on $[0, H]$, for every $u_0 \in \mathcal{R}$. [conventional; identity (35), its bound and both examples Lean]

Proof. For $Q_0 - D \leq Q \leq Q_0$,

$$0 \leq s_0 - s_{\mathrm{eff}}(Q) = \frac{s_0 K(Q_0 - Q)}{Q_0(K+Q)} \leq \frac{s_0 K D}{Q_0(K + Q_0 - D)} \leq \delta. \tag{35}$$

The proof of Theorem 5.1 now applies verbatim on the tube $Q_0 - 16t \leq Q \leq Q_0$: the source stays in the checked band, and $R \leq (3/25)\,375 \cdot 30/87.3 < 16$ on the lower donor face. Equality in (34) is allowed because the face inequalities (28) are strict. The quadratic form follows by clearing denominators in (34), whose positive root exceeds D. $\square$

For $H = 251/250$ (so $D = 16.064\ \mu$M), two exact instances are

$$\begin{aligned} (Q_0, K) = (200\ \mu\mathrm{M},\ .01\ \mu\mathrm{M}): &\qquad s_0 - s_{\mathrm{eff}} \leq \tfrac{753}{1437078125} \approx 5.24 \cdot 10^{-7}, \\ (Q_0, K) = (1500\ \mu\mathrm{M},\ 1\ \mu\mathrm{M}): &\qquad s_0 - s_{\mathrm{eff}} \leq \tfrac{502}{580053125} \approx 8.65 \cdot 10^{-7}. \end{aligned}$$

These stocks are 9638 and 1285 times smaller than under the linear law, for all $u_0 \in \mathcal{R}$ and with the refined floors 10.35 and 4.01. When the square root dominates, (34) scales like $\sqrt{s_0 KD/\delta}$; the exact formula, not this approximation, should be used for long horizons. The values of K are not fitted or attributed to a donor–enzyme pair; an experimental realization would need capacity, affinity and stock to be separately identifiable. The corollary also shows how a dynamic compatibility theorem is reused: a new supply is admissible when its *coupled* depletion keeps it inside a proved operating envelope, and the envelope must be verified on the coupled tube.

5.3 Linear donor law: the depleted trajectory

We return to the unchanged law (30) and certify the trajectory that actually occurs, letting the source fall far outside the band.

Theorem 5.3 (Buffered finite mission)**.** *Let $H = 1.004$, let $u_0 \in \mathcal{P}_\kappa$ with $\kappa = 10^{-3}$ as in Corollary 4.3 (so $H_T(u_0) < 4$), and let $|Q(0) - \frac{3}{25}V| \leq 10^{-3}\ \mu$M in (30).*

(a) $V = 1000$, $Q_0 = 120$ μM: success. *There is a unique solution on* $[0, H]$*; it is physical, and*

$$H_G \geq 10.3499 \quad (0 \leq t \leq H), \qquad H_T \geq 4.034 \quad (\tau \leq t \leq H), \qquad \int_0^H [4 - H_T]_+ \, \mathrm{d}t \leq 1.1 \cdot 10^{-4}.$$

The donor spent on $[\tau, H]$ *lies in* $[14.034, 14.045]$ μM *and the final source scale in* $[.10589, .10591]$*. For all* $t \geq .434$ *one has* $s(t) = Q(t)/V < .1137126 \leq s_{\min}$*. The storage drawdown satisfies*

$$\int_0^H (H_G + H_T) \, \mathrm{d}t - \int_0^H R_{Q/V}(x) \, \mathrm{d}t = W(u(0)) - W(u(H)) \in [.700, .751] \ \mu\mathrm{M}. \tag{36}$$

(b) $V = 500$, $Q_0 = 60$ μM: failure. *There is a unique physical solution on* $[0, H]$*, and* $H_T(u(H)) \leq 3.9132 < 4$*. The mission of Definition 1.1 with* $\tau = .004$ *and* $H = 1.004$ *fails from every preparation in* $\mathcal{P}_\kappa$*.*

[exact rational tube certificates, replayable; not kernel-checked]

The proof is the certificate of Appendix A: a piecewise-affine reference $(\bar{c}(t), \bar{C}(t))$ for the state and the spent donor, affine radii, and 9 strict face inequalities on each of 1206 time steps, which verify the hypotheses of Lemma 4.1 for the nine-state system uniformly in time. Every inequality is evaluated in exact rational arithmetic.

Three points deserve emphasis. First, the certified sufficient stock falls from 1.9 M to 120 μM under the *same* law, a factor 16,064, and the sufficient and insufficient stocks differ by a factor of two. A numerical bisection from p places the threshold near 70.9 μM, and a numerical solution with $Q_0 = 120$ μM holds the Trx quota until about 1.387 s; these two numbers are observations. Second, the success in (a) is not a refinement of the stationary picture but runs against it: for the last 57% of the mission the source is below $s_{\min}$, where by Theorem 3.1 *no* stationary state meets both quotas, and yet both quotas are met with margin. By (36), about 0.7 μM of the 14.8 μM of delivered service is paid from stored reduced carrier rather than from regeneration. A rule that applies the stationary threshold at each instant would reject a mission that provably succeeds. Third, the old storage range (26) is not used: it belongs to $\mathcal{S}$, which the depleted trajectory leaves, and the accounts in (a) are computed on the actual tube.

We could not extend (a) to all of $\mathcal{R}$, and the obstruction is physical rather than technical: $\mathcal{R}$ contains states whose nominal floor is only $H_T \geq 4.01$, carried by slow coordinates that do not relax within a second, and a 12% fall of the source removes that margin.

6 A worked comparison

Table 3 places the three questions side by side, for one chemistry, one pair of quotas and common units.

The first two rows have the same stationary description and recovery times that differ by more than four orders of magnitude; the difference is retained private state. The third row is a case where the stationary threshold is pessimistic. In an assay, x corresponds to measured NADPH, service to branch peroxide-reduction flux, and w to the sulfinylated peroxiredoxin fraction; the quotas are design values and are not thresholds of cellular health.

7 Discussion and limits

What a modeler needs. To guarantee joint service one needs more than branch response curves. For *capacity*, the curves and the source suffice [7]. For *recovery*, one needs a bound on the initial private state in the directions that do not relax before the deadline, and the

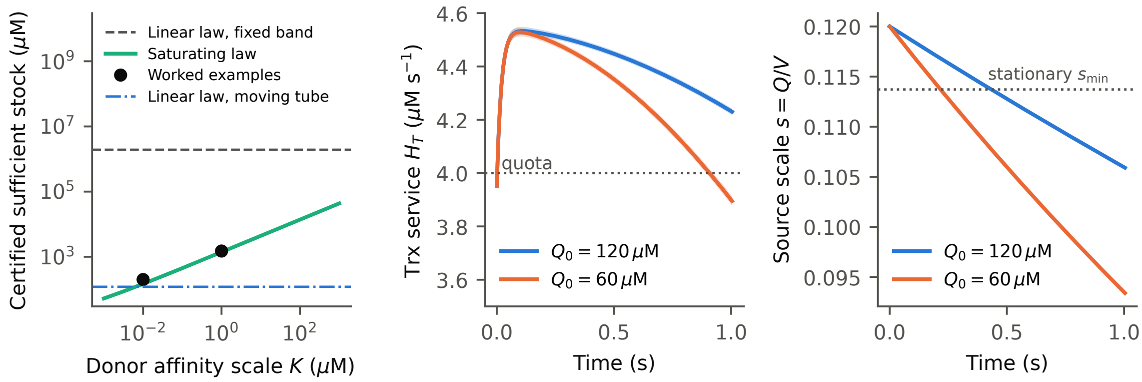

Figure 4: Finite donor. Left: certified sufficient stock for the one-second mission: the saturating-law curve (34) against affinity scale, and the two linear-law results as horizontal lines; curves for different laws are not one "optimal stock" curve. Middle: H_T for the linear law with $Q_0 = 120$ and 60 μM; bands are exact enclosures over $\mathcal{P}_\kappa$ and donor tolerance, curves are the certificate references. Right: the source scale falls below the stationary threshold $s_{\min}$ at $t \approx .43$ s for $Q_0 = 120$ μM while the quota continues to hold.

Table 3: What limited information predicts and what the full-state analysis shows. All rows use quotas $(10, 4)$ μM s^{-1}.

Situation	From stationary data	Full-state conclusion
Adequate source $s = .12$; small oxidation defect ($\mathcal{P}_\kappa$, $H_T(u_0) < 4$)	Compatible ($s > s_{\min}$); no deadline	Both quotas from 4 ms on, forever; Trx shortfall $\leq 1.1 \cdot 10^{-4}$ μM; regeneration $\geq 14.36T - .1$ on later windows (Thm. 4.2, Cor. 4.3)
Same stationary responses; 95% hyperoxidized pool; repair scale σ	Compatible, identically for all σ	No service before $70.9/\sigma$ s at any source strength (Prop. 3.4); observed ≈ 537 s at $\sigma = 1$
Linear donor, $Q_0 = 120$ μM, one-second mission	Incompatible for $t \geq .43$ s ($s < s_{\min}$)	Mission succeeds with $H_T \geq 4.034$; 0.70–0.75 μM drawn from storage (Thm. 5.3a)
Linear donor, $Q_0 = 60$ μM	Incompatible for $t \geq .22$ s	Mission fails: $H_T(u(H)) \leq 3.913$ (Thm. 5.3b)
Any finite donor, indefinite service	not addressed	Impossible (Prop. 2.1)

kinetics of the slowest process that gates service; Proposition 3.3 shows this is unavoidable. For *duration*, one needs the supply law and the stored account W, and the verdict must be read from the depleted trajectory: a budget computed on an undepleted nominal run is a necessary condition only.

Compositional use. For modules $\dot z_i = f_i(x, z_i)$ with outputs H_i, demands D_i and $\dot x = J - \sum_i D_i$, Lemma 4.1 can be applied blockwise: bound the private faces uniformly over a resource interval, and bound the *loaded* resource faces using those same private sets, all in one first-exit argument. Assuming a favorable resource trace and then presenting it as a conclusion about the coupled system would be circular. In the present model interval bounds on the private blocks destroy signed cancellations between the x, z_T and v directions, which is why the certificates above use joint modal coordinates.

Limits. The positive theorems are sufficient certificates on explicit regions. They prove no global basin, no optimal deadline, no minimal stock for the linear law (only the bracket 60–120 μM), and nothing about the source-.1 equilibrium or heavily hyperoxidized preparations beyond the necessary delay. The 0.1% box is small in absolute terms. The donor models make only regeneration finite; peroxide and the repair donor remain maintained, so the model is not thermodynamically closed. Parameters are declared, not calibrated. The full evolution statements of Theorems 4.4 and 5.1 and Corollary 5.2 are conventional proofs resting on kernel-checked inequalities, and Theorem 5.3 and the finite-horizon part of Corollary 4.3 rest on exact arithmetic that is replayable but not kernel-checked; a reusable formal nonautonomous comparison lemma would close the first gap.

A The moving-tube certificate

Data. A certificate is a list of rational times $0 = t_0 < \cdots < t_n = H$ containing τ, rational reference points $\bar{c}_k \in \mathbb{R}^8$ and $\bar{C}_k \in \mathbb{R}$ for the state and the spent donor $C = Q_0 - Q$, and rational radii $q_k \in \mathbb{R}^8_{>0}$, $q_k^C > 0$. References and radii are affine between grid times. The tube is

$$u(t) \in \bar{c}(t) + M\,\mathcal{B}_0(q(t)), \qquad |C(t) - \bar{C}(t)| \le q^C(t),$$

with $\mathcal{B}_0(q) = \{a : |a_j| \le q_j\}$ and the *same* M as in Section 4. The source is $s = \frac{3}{25} - C/V$, with $1/V = 0$ for the maintained source. Reference points are a numerical solution rounded to 13 decimals; they carry no evidential weight, because their residual is bounded exactly. Initial data are $\bar{c}_0 = p$, $q_{0,i} \ge \kappa \sum_j |(M^{-1})_{ij}| p_j$, $\bar{C}_0 = 0$ and $q_0^C = 10^{-3}$.

One step. Fix a step of length h, and let c_m, C_m be midpoints, Δ, Δ_C the increments, $s_m = \frac{3}{25} - C_m/V$, $J = DF_{s_m,1}(c_m)$, $A = M^{-1}JM$, $\hat{q} = \max(q_k, q_{k+1})$ and $\hat{q}^C$ likewise. In the step, $u = c_m + d$ with $d = \theta\Delta + Ma$, $|\theta| \le \frac{1}{2}$, so $|d_j| \le \hat{d}_j = \sum_k |M_{jk}|\hat{q}_k + \frac{1}{2}|\Delta_j|$, and $|s - s_m| \le \hat{\varsigma} = (\hat{q}^C + \frac{1}{2}|\Delta_C|)/V$. With $a = M^{-1}(u - \bar{c}(t))$,

$$\dot{a} = M^{-1}(F_{s_m,1}(c_m) - \Delta/h) + \theta M^{-1}J\Delta + Aa + M^{-1}N(d) + (s - s_m)R_1(x)M^{-1}e_x,$$

where N is the exact second-order remainder of Section 4.3 with c_m, s_m in place of $c, \frac{3}{25}$. Hence on the face $a_i = \eta q_i(t)$,

$$\begin{aligned} \eta\dot{a}_i \le A_{ii}q_i(t) + \sum_{j\neq i} |A_{ij}|\hat{q}_j + \big|[M^{-1}(F_{s_m,1}(c_m) - \Delta/h)]_i\big| \\ + \tfrac{1}{2}\big|[M^{-1}J\Delta]_i\big| + n_i(\hat{d}) + \hat{\varsigma}\,R_1(c_{m,0} - \hat{d}_0)\,|(M^{-1})_{i0}|, \end{aligned} \tag{37}$$

using that R_1 is positive and decreasing. The first term is affine in t and is replaced by its larger endpoint value. The check is that the right side is strictly less than $(q_{k+1,i} - q_{k,i})/h$. Radii may grow; slow coordinates are allowed to, which is what a finite horizon permits and an invariant set does not. For the donor faces,

$$\big|(C - \bar{C})'\big| \le \big|s_m R_1(c_{m,0}) - \Delta_C/h\big| + s_m \frac{375 \cdot 57.3}{(87.3 - c_{m,0} - \hat{d}_0)^2}\,\hat{d}_0 + \hat{\varsigma}\,R_1(c_{m,0} - \hat{d}_0) < \frac{q_{k+1}^C - q_k^C}{h}.$$

The thirteen physical margins of (6) are checked to be positive on the whole slice, and the floors (25) are evaluated with $c_m, \hat{d}$. Lemma 4.1 on each step, joined at grid times, and compact continuation give existence, uniqueness and the enclosure. Accounts use endpoint slices: spent donor is $\bar{C} \pm q^C$, and $W \in 234.43 + \ell\bar{c} \pm \sum_i |(\ell M)_i| q_i$.

Outcome. All three certificates use 1206 steps, geometric from $2 \cdot 10^{-6}$ s and then uniform. Table 4 lists the verified quantities. The smallest face margin is positive in each case; it is tiny by design, since radii are chosen just above what the inequalities need.

Table 4: Verified outcomes of the tube certificates (exact values rounded outward).

	maintained source	$Q_0 = 120\ \mu$M	$Q_0 = 60\ \mu$M
min H_G floor on $[\tau, H]$	10.34995	10.34990	10.34963
min H_T floor on $[\tau, H]$	4.03438	4.03438	3.87684
H_T enclosure at H	$[4.5326, 4.5484]$	$[4.2182, 4.2421]$	$[3.8770, 3.9132]$
$H_T \geq 4$ certified from (ms)	2.75	2.75	–
Trx shortfall bound (μM)	$1.017 \cdot 10^{-4}$	$1.018 \cdot 10^{-4}$	$1.018 \cdot 10^{-4}$
Regeneration on $[0, H]$ (μM)	$[14.947, 14.953]$	$[14.095, 14.104]$	$[13.297, 13.310]$
Final source scale	.12	$[.105896, .105905]$	$[.093381, .093405]$
$W(u(0)) - W(u(H))$ (μM)	$[-.035, .013]$	$[.700, .751]$	$[1.374, 1.429]$
Slice inside $\mathcal{S}$ from	.2477 s	never	never

B Evidence and reproduction

Formal evidence. The Lean development uses Lean 4 [10], version 4.30.0, and Mathlib [22] at revision

`c5ea00351c28e24afc9f0f84379aa41082b1188f`.

Theorem 4.2 is the declaration

`DynamicSharedResource.Certificate.fast_joint_service`

in `proofs/DynamicSharedResource/OperationalMain.lean`, compiled with warnings as errors together with its 27 local dependencies. The reported axioms are `Classical.choice`, `Quot.sound` and `propext`; no `sorry` and no native-computation oracle is used. The rational data of M, M^{-1}, A, b and n are Lean definitions in `Data.lean`, and the identities relating them to the literal field are proved, so neither the generator script nor a file hash carries mathematical weight. `SourcePerturbation.lean` proves $0 \leq R_1 \leq 130$, the modal source difference and (28). `RepairObstruction.lean` proves Proposition 3.2, $\dot{w} \geq -.003\sigma w$ and the integrating-factor inequality. `SaturatingDonorBounds.lean`, new with this paper, proves (35), band membership, and both numerical instances. The flow statements of Theorems 4.4 and 5.1, Corollary 5.2, Propositions 2.1 and 3.3, the logarithmic conclusion of Proposition 3.4, and Lemma 4.1 with a moving center are conventional.

Exact-arithmetic evidence. The rational certificate `data/modal_box.json` has SHA-256

`09ff83c30f24cd6b86fea53cad2103dfd7f58cd2300facdcf7a62f962c88c3f8`.

The tube certificates are `data/tube_nominal.json`, `data/tube_Q120.json` and `data/tube_Q60.json`. The command

```
python scripts/depleted_tube.py verify tube_Q120
```

replays every inequality of Appendix A with Python integers and `Fractions` only; no floating-point number enters a verified comparison, and the checker also verifies $M^{-1}M = I$. Floating point is used only by the `build` mode to propose references and radii. `scripts/preparation_geometry.py` computes the quantities of Section 4.4 exactly.

Numerical illustrations. Trajectory curves, the repair-clock crossings, the threshold near 70.9 μM and the 1.387 s lifetime are SciPy Radau computations with relative tolerance at most $2 \cdot 10^{-10}$. They illustrate and are never used as proof. `scripts/make_figures.py` regenerates all figures and writes their data.

Building the manuscript. From the source directory:

```
pdflatex main.tex
bibtex main
pdflatex main.tex
pdflatex main.tex
```

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