Reliable productive operation of coupled autocatalytic reactors under mechanistic refinement

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Abstract

When does a chemical reactor retain useful operation after it is coupled to other reactors and an effective reaction is replaced by an explicit intermediate mechanism? For a specified reversible autocatalytic exporter, we prove a finite-mission guarantee that includes recovery after molecular withdrawal, loss and refill, collected product, food input and gross reaction service. On a symmetric exchange graph with n reactors, weighted degree at most Δ and integer count scale $V \geq 10^4$, the seven-species refinement completes m cycles with probability at least $\max\{0, 1 - 29nm\exp[-V/(2 \cdot 10^{12}(1+\Delta)^3)]\}$. Policies may use the complete history of returned states and physical counters. An exact material-forced reduction of the six-species donor explains which information can be removed; an explicit witness shows why simply deleting the intermediate is not an exact reduction. The proof controls material, stock and phase separately, then composes from actual random endpoints. An augmented-inventory identity shows that every successful mission of at least 55 cycles necessarily produces net new inventory beyond depletion of its initial supply. A connected two-reactor, 100-cycle instance has certified success probability above 99%. The refined operational results have Lean-checked proofs; the donor theorem and the additional accounting consequences are proved conventionally. The count scale is sufficient, not optimized or experimentally calibrated.

1 Introduction

Structural autocatalysis and reliable production are different mathematical questions. The RAF framework identifies reaction sets that are reflexively autocatalytic and food-generated: their catalysts are available from the food set or the set's products, and their reactants can be built from food using the set's reactions [5]. Such a structural criterion does not by itself specify rates, washout, a withdrawal protocol or the probability of meeting a product quota. Here autocatalysis has an explicit kinetic realization: the productive route $X + U \rightleftarrows C_1$, $C_1 + W \rightleftarrows C_2$, $C_2 \rightleftarrows Z$, $Z \rightleftarrows 2X$ converts one X into two while using food. We analyze this specified exporter; we do not infer operation from a generic RAF classification or identify a particular laboratory chemistry.

Two modifications make operation more demanding. Conservative transport preserves global material but creates local fluctuations and dependence. Inserting an intermediate stores material, introduces a washout channel, and separates upstream from downstream conversion. Existing inheritance results address related but distinct properties. Feliu and Wiuf relate models with intermediates through steady-state parameter matching [3]; Banaji proves preservation of the capacity for nondegenerate equilibria and periodic orbits under suitable reaction splitting [1]. Cappelletti and Wiuf study elimination of intermediates in multiscale stochastic networks [2]. These results do not directly give the fixed-parameter, intervention-dependent output and resource event considered here.

Our contribution is a source-specific guarantee for useful operation under both modifications. Success is defined by actual collected molecules and the actual state returned after each cycle, rather than by closeness of all trajectories to a donor model. The principal theorem treats the literal seven-species jump process with an intermediate lifetime parameter fixed at $\theta = 0.01$. A separate local deterministic theorem allows $0 < \theta \leq 0.01$. The six-species donor provides an exact material-forced reduction and a conventional network reliability theorem. Neither theorem is obtained by assuming the intermediate vanishes. The improved exponential constant for the refined source comes from a less restrictive internal error budget; it is not a comparative claim that the refined chemistry is intrinsically more reliable.

The argument combines positive material balances, a productive-stock barrier, a short phase comparison and exponential martingale bounds. The last belong to the general time-uniform concentration framework [6]; we give the bound and the source estimates used here. A pathwise inventory telescope then distinguishes sustained net synthesis from spending initially carried stock. The paper includes the readable proof and the quantitative estimates required for its conclusions; the accompanying proof map separates formal checks, conventional deductions and numerical illustrations.

2 The literal chemistry and intervention

Write the concentration vector as $c = (u, w, x, c_1, c_2, z, h) \in \mathbb{R}^7_+$, for species $(U, W, X, C_1, C_2, Z, D)$, and molecular counts as $N = Vc$. The rows of Table 1, indexed $0, \ldots, 6$, specify reversible pairs; coefficients multiply the usual concentration monomial. Every species has unit washout, and U, W each have unit feed. Let $\epsilon_0 = 1/(5 \cdot 10^8)$, $\eta_0 = 1/(8 \cdot 10^9)$, $19 \leq r \leq 21$, $1/50 \leq d \leq 1/25$, and $\beta = 1/100$.

Pair	Forward coefficient	Reverse coefficient
$U + W \rightleftarrows X$	$1/(5 \cdot 10^8)$	$1/(5 \cdot 10^9)$
$X + U \rightleftarrows C_1$	20	20
$C_1 + W \rightleftarrows C_2$	20	20
$C_2 \rightleftarrows Z$	20	2
$Z \rightleftarrows 2X$	r	r
$X \rightleftarrows D$	$d(1+\beta)$	$d\beta/\theta$
$D \rightleftarrows U + W$	d/θ	$d\eta_0(1+\beta)/\beta$

Table 1: The seven chemical pairs. Two feeds and seven washouts give 23 local labels. Gross service counts both directions of $X \rightleftarrows D$.

The deterministic vector field is $f(c) = (1, 1, 0, 0, 0, 0, 0) - c + \sum_{j=0}^{6} \nu_j j_j(c)$, where ν_j is the product-minus-reactant vector and j_j the net flux of row j. In particular, the two new net fluxes are

$$j_5 = d(1+\beta)x - d\beta h/\theta, \qquad j_6 = dh/\theta - d\eta_0(1+\beta)uw/\beta. \tag{1}$$

The intermediate is retained as a state variable at all times, including between cycles. Define augmented materials, productive stock and product inventory by

$$A = u + x + 2c_1 + 2c_2 + 2z + h, \tag{2}$$
$$B = w + x + c_1 + 2c_2 + 2z + h, \tag{3}$$
$$Y = x + \tfrac{9}{8}c_1 + \tfrac{7}{5}c_2 + \tfrac{9}{5}z, \tag{4}$$
$$I = x + c_1 + c_2 + 2z. \tag{5}$$

For nonnegative states, $I \geq 5Y/7$. The gross service rate is

$$g(c) = d(1+\beta)x + d\beta h/\theta. \tag{6}$$

It is a sum of directional rates, not their net difference.

An intervention chooses $q \in [1/4, 3/4]$, survival factors $\ell_s \in [49/50, 1]$ for all seven species, and food errors $e_U, e_W \in [-1/200, 1/200]$. It maps c_s to $q\ell_s c_s$ and then adds $1 - q + e_U$ to U and $1 - q + e_W$ to W. In the count process each molecule is independently categorized, conditional on the action and current state, as withdrawn, lost or retained with probabilities $1-q$, $q(1-\ell_s)$ and $q\ell_s$; the refill is the actual integer floor $\lfloor V(1 - q + e_s) \rfloor$. Choices may differ by compartment and depend on the available past history.

Definition 1. For $\theta > 0$, $c \in \mathsf{Ready}(\theta)$ if $c \geq 0$ and

$$\frac{159}{160} \leq A, B \leq \frac{161}{160}, \qquad Y \geq \frac{1}{20}, \qquad h \leq \frac{1101}{1000}\theta. \tag{7}$$

One cycle consists of the intervention followed by four time units of flow. The product collection window is $(3, 4]$; the food and service accounting window is $(0, 4]$, with refill included separately in food.

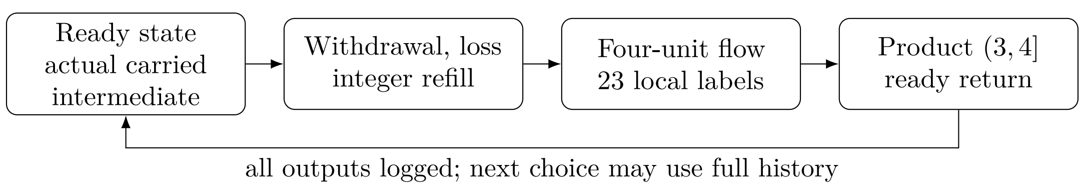

Figure 1: The physical cycle. Product marks are existing washout events. Repetition uses the actual returned state, and failed outcomes remain in the probability law.

2.1 The donor and the common operating contract

The donor uses the first five pairs of Table 1, but replaces its last two pairs by $X + F \rightleftarrows U + W + P$ with forward and reverse rates dx and $d\eta_0 uw$. The external species F, P have unit maintained activity and are not internal material coordinates. A compatible bookkeeping realization of the split rates is $X + F \rightleftarrows D \rightleftarrows U + W + P$; no chemical potentials or energy costs are assigned here. The donor has 20 local labels: twelve directed chemical channels, two feeds and six washouts. Define its materials $A_0 = u + x + 2c_1 + 2c_2 + 2z$, $B_0 = w + x + c_1 + 2c_2 + 2z$. Its ready set has the same material interval and stock floor as (7), with A_0, B_0 replacing A, B and no intermediate constraint. Its service counts both directions of its single driven pair. All other pulse, collection and food conventions are the same. For either process, jumps at a fixed positive time have probability zero, so the donor convention $[3, 4]$ and the adopted $(3, 4]$ define the same event almost surely. The latter convention is used for the literal refined marked law.

3 Operational theorems

Theorem 2 (Local deterministic repeated operation). *Fix $\beta = 1/100$, $0 < \theta \le 1/100$ and the stated r, d ranges. From any state in* $\mathsf{Ready}(\theta)$*, every sequence of admissible interventions, chosen from the cycle index and current state, admits successive nonnegative global solutions of the literal seven-species ODE. Every four-unit endpoint is in* $\mathsf{Ready}(\theta)$*. In each cycle, stock is at least $1/20$ for $t \ge 5/2$, and*

$$\int_3^4 I \,\mathrm{d}t \ge \frac{1}{28}, \qquad \int_3^4 x \,\mathrm{d}t \ge \frac{1}{540}, \qquad \int_0^4 g \,\mathrm{d}t \le \frac{112201}{625000}. \tag{8}$$

Each food account, including refill, is at most $951/200$ per cycle. Consequently all these inequalities sum over every finite prefix of cycles.

For the stochastic statement take $\theta = \beta = 1/100$. There are $n > 0$ compartments, with rates r_i, d_i in the same ranges. A nonnegative symmetric matrix k with $k_{ii} = 0$ acts on every species by molecule exchange: a molecule of species s moves from i to j at total rate $k_{ij}N_{i,s}$. Assume $\sum_j k_{ij} \le \Delta$, $\Delta \ge 0$. Write

$$(\mathcal{D}a)_i = \sum_j k_{ij}(a_j - a_i).$$

The exchange is nonselective and the count scale is the same at every node. For a reaction with consumption vector a, production vector b and coefficient κ, its propensity at a count state is

$$\lambda(N) = \kappa V \prod_s \frac{(N_s)_{a_s}}{V^{a_s}}, \qquad (N)_a = N(N-1)\cdots(N-a+1). \tag{9}$$

The jump is $N - a + b$ when enabled; disabled channels have rate zero. In particular, $2X \to Z$ has propensity $rN_X(N_X - 1)/V$, with no factorial divisor. Each food feed has rate V, and each washout label has rate N_s. These propensities define the continuous-time Markov reaction law through exponential holding times and rate-proportional reaction choices [4].

Let Q_I, Q_X be the natural-number product marks in $(3, 4]$. Washout of X, C_1, C_2, Z contributes respectively $1, 1, 1, 2$ to Q_I; only washout of X contributes to Q_X. Let F_U, F_W include continuous food arrivals and pulse refill. Let G count both directions of $X \rightleftarrows D$ in $(0, 4]$. Intermediate washout is not a product mark.

Theorem 3 (Finite-count mission). *Let $m \in \mathbb{N}$, $V \in \mathbb{N}$, $V \ge 10000$, and let every initial compartment be ready after normalization by V. Under the unrestricted literal pulse-plus-flow law, every admissible controller on the history of returned states and counters satisfies*

$$\mathbb{P}(\textit{all } m \textit{ cycles succeed}) \;\ge\; \max\left\{0, 1 - 29nm \exp\left(-\frac{V}{2 \cdot 10^{12}(1+\Delta)^3}\right)\right\}. \tag{10}$$

Success means that, in every compartment and cycle, the actual endpoint is in $\mathsf{Ready}(1/100)$ *and*

$$Q_I \ge \lceil V/56 \rceil, \quad Q_X \ge \lceil V/1080 \rceil, \quad F_U, F_W \le 5V, \quad G \le \lfloor V/5 \rfloor. \tag{11}$$

The same bound holds on any measurable history space whose next observed output has the literal pulse-plus-flow conditional law, whose current counts equal that observed return, and whose log appends the actual output.

The history assumptions in the last sentence specify a model interface; they do not assume a success probability. A normalized history kernel for arbitrary policies on all previous returned states and counters is explicitly constructed. Richer observations are covered when their conditional law satisfies the stated interface. No independence between nodes or successive cycles is required.

Theorem 4 (Coupled donor operation)**.** *For the six-species donor, $V \geq 10000$, the stated graph and rate assumptions, and initial readiness at every node, all m cycles meet the common integer contract with probability at least*

$$\max\{0, 1 - 27nm\exp[-V/(2 \cdot 10^{14}(1+\Delta)^3)]\}.$$

The controller may use the actual joint past. This theorem has a conventional proof; Appendix B supplies its source estimates and probability budget.

Corollary 5 (Sizing and yield)**.** *For $m \geq 1$ and $0 < \delta < 1$, it suffices that*

$$V \geq \max\left\{10000,\ 2 \cdot 10^{12}(1+\Delta)^3 \log \frac{29nm}{\delta}\right\} \tag{12}$$

to guarantee mission probability at least $1-\delta$. On the success event, the division-free food-to-product inequalities

$$F_U + F_W \leq 560 Q_I, \qquad F_U + F_W \leq 10800 Q_X \tag{13}$$

hold per compartment per cycle, and hence after summing over any successful mission.

Proof. Exponentiation of the sizing inequality bounds the error term in (10) by δ. Also $F_U + F_W \leq 10V$, while $Q_I \geq V/56$ and $Q_X \geq V/1080$. These comparisons do not divide by a random output. ☐

For a concrete connected example take two nodes, $k_{12} = k_{21} = 1$, zero diagonal, and $V = 224000000000000$. Choose initial normalized states

$$c^{(1)} = (19/20, 19/20, 1/20, 0, 0, 0, 0), \quad c^{(2)} = (13/14, 13/14, 0, 0, 1/28, 0, 0). \tag{14}$$

Both count vectors are integral at this V and ready. For $m = 100$, the proved lower bound is $1 - 5800e^{-14} > 0.99$. A rational Taylor certificate $e^{14} \geq 600000$ is sufficient to verify the strict inequality. The scale is sufficient, not claimed minimal.

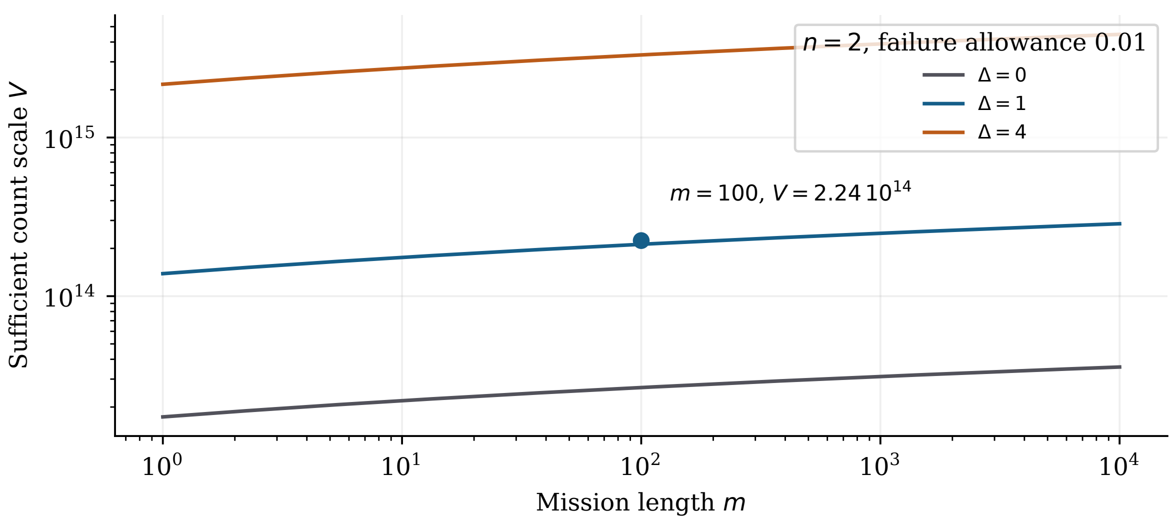

Figure 2: The certified sufficient scale from Corollary 5. The dot is the connected 100-cycle instance. Curves describe a theorem bound, not empirical failure probabilities or calibrated molecule requirements.

4 Material closure and the role of the intermediate

Proposition 6 (Exact material-forced donor reduction)**.** *On each deterministic flow segment, the donor is equivalent to the four phase equations for* (x, c_1, c_2, z) *supplied with*

$$A_0(t) = \mathbf{1} + e^{-t} e^{t\mathcal{D}}(A_0(0) - \mathbf{1}), \quad B_0(t) = \mathbf{1} + e^{-t} e^{t\mathcal{D}}(B_0(0) - \mathbf{1}), \tag{15}$$

$$u_i = A_{0i} - x_i - 2c_{1i} - 2c_{2i} - 2z_i, \quad w_i = B_{0i} - x_i - c_{1i} - 2c_{2i} - 2z_i. \tag{16}$$

The domain is that on which the phases and both reconstructed foods are nonnegative. Material initial data and their actual deterministic pulse updates are retained.

Proof. Dotting each chemical increment with either material weight gives zero; feed, washout and exchange therefore give $A_0' = \mathbf{1} - A_0 + \mathcal{D}A_0$ and the same equation for B_0. Solving gives (15). The four phase equations are

$$x' = -x + j_0 - j_1 + 2j_4 - j_5 + \mathcal{D}x, \quad c_1' = -c_1 + j_1 - j_2 + \mathcal{D}c_1,$$

$$c_2' = -c_2 + j_2 - j_3 + \mathcal{D}c_2, \qquad z' = -z + j_3 - j_4 + \mathcal{D}z,$$

where $j_0 = \epsilon_0 uw - \epsilon_0 x/10$, $j_1 = 20xu - 20c_1$, $j_2 = 20c_1 w - 20c_2$, $j_3 = 20c_2 - 2z$, $j_4 = r(z - x^2)$ and $j_5 = d(x - \eta_0 uw)$. Substitution proves the forward implication. Conversely, differentiating (16) recovers the two food equations, since fixed species-linear combinations commute with $\mathcal{D}$. Polynomial local uniqueness identifies both constructions. Positivity and bounded material ensure global existence. At each pulse retain each species with its own $q\ell_s$, add the food doses, and recompute the materials from these values; this completes the segmentwise equivalence. □

In particular, resetting $A_0 = B_0 = 1$ after a pulse generally changes the equations. For count paths these material equations describe drift only: local transfer and reaction histories cannot be replaced by the deterministic formula (15). In the refinement, $A = A_0 + h$ and $B = B_0 + h$ obey the same linear drift identities.

Proposition 7 (Storage, loss and nonclosure)**.** *In a refined reactor,*

$$j_5 - j_6 = h' + h - \mathcal{D}h.$$

Thus the global integrated upstream-minus-downstream conversion equals the change in stored intermediate plus its washout, with pulse removals added across interventions. Deleting h alone gives neither an exact autonomous deterministic system nor an exact Markov projection valid for all ready initial states.

Proof. The intermediate equation is $h' = j_5 - j_6 - h + \mathcal{D}h$; global exchange sums to zero by symmetry. Integration on flow segments and telescoping across pulses prove the balance. For nonclosure, compare the ready states

$$(19/20, 19/20, 1/20, 0, 0, 0, 0), \qquad (19/20, 19/20, 1/20, 0, 0, 0, 1/200).$$

At $\theta = \beta = 1/100$ both have $Y = 1/20$ and materials respectively 1 and 1.005. Their six retained coordinates agree, but the intermediate contributes $d\beta h/\theta$ to x' and dh/θ to each food derivative. These contributions differ. For V divisible by 200, both are integer states and have distinct total rates to the same projected food-increment state. The generator therefore cannot act on the retained coordinates alone for all such initial states. □

At fixed x, u, w without exchange, the algebraic filter equilibrium is

$$h_{\mathrm{eq}} = \frac{d(1+\beta)\theta}{\theta + d(1+\beta)} \left(x + \frac{\eta_0}{\beta} uw \right).$$

Substitution into j_5 gives

$$j_{5,\mathrm{eq}} = \frac{d(1+\beta)(\theta+d)}{\theta + d(1+\beta)} x - \frac{d^2 \eta_0 (1+\beta)}{\theta + d(1+\beta)} uw.$$

Its algebraic limit as $\theta \downarrow 0$ is the donor rate $d(x - \eta_0 uw)$. This explains rate matching, but asserts no trajectory or jump-law convergence. At finite θ, $j_{5,\mathrm{eq}} - j_{6,\mathrm{eq}} = h_{\mathrm{eq}}$ remains a real loss.

5 Proof of productive operation

5.1 Source identities and deterministic return

For a single reactor without exchange, the stoichiometry gives $A' = 1 - A$ and $B' = 1 - B$ exactly. Moreover,

$$h' = d(1+\beta)(x + (\eta_0/\beta)uw) - \left(1 + \frac{d(1+\beta)}{\theta} \right) h. \tag{17}$$

Thus h is a dissipative filter with explicit forcing, rather than an algebraically eliminated species. Positive material weights bound every coordinate and prove global existence on the positive orthant. Admissible pulses retain at least 49/4000 stock; the material and intermediate bounds then imply a positive free-material floor. The exact stock drift is bounded below by $(2/3)Y$ in the required low-stock region. A scalar comparison restores $Y \geq 1/20$ by time 5/2. The four productive phases transfer a fixed fraction of this stock into x during the collection interval. Integration gives (8), while (6) and the intermediate envelope give the service budget. Material return and the filter bound put the endpoint back in (7). Applying this uniform returning-cycle construction recursively proves Theorem 2; summation gives its finite-prefix accounts. This argument uses the actual refined vector field for every fixed admissible θ and does not replace it by its six-species donor.

5.2 Literal count law and continuous compensation

The propensity formula (9) agrees with the chemical drift except for the exact falling-factorial correction

$$f_V(c) = f(c) + (0, 0, 2rx/V, 0, 0, -rx/V, 0). \tag{18}$$

It leaves both material drifts unchanged and adds $rx/(5V) \geq 0$ to the stock drift. The chronological exponential-clock law is normalized and nonexplosive: positive material weights bound rates on finite material sublevels, and only the two feed channels increase total material. The corresponding food-arrival bound removes the finite-sublevel localization. Count and marked evaluations at a fixed time are measurable, including jumps exactly at that time.

For a coordinate, subtract its continuous drift integral from its normalized count path. Stop at the first exit of any material from $[0.9, 1.1]$, including that jump. The local propensity sum divided by V is below 200 on this corridor; coordinate jumps are at most $2/V$, and incoming plus outgoing exchange has squared-jump rate at most $2.2\Delta/V$. The four-time coordinate bracket is therefore at most $3200(1+\Delta)/V$. The exponential supermartingale argument in Appendix B, or its censored exponential-clock implementation, gives give, over the four-unit interval, the tail

$$2 \exp\{-\vartheta a + \vartheta^2 (800(1+\Delta)/V)4\} \tag{19}$$

for deviation at least $a + 2/V$, whenever $2\vartheta/V \le 1$. The supremum includes unfinished holding intervals and the exit jump. This is essential: a bound at completed jumps alone would not control the deadline.

Put $q_\Delta = 1 + \Delta$ and choose

$$\varepsilon = \frac{1}{10000 q_\Delta}, \qquad a = \frac{19}{20}\varepsilon, \qquad \vartheta = \frac{Va}{6400 q_\Delta}. \tag{20}$$

When $V \ge 40/\varepsilon$, the jump reserve is at most $\varepsilon/20$, and (19) is at most

$$2\exp\left\{-\frac{361}{5120000}\frac{V\varepsilon^2}{q_\Delta}\right\} \le 2\exp\left\{-\frac{V}{2\cdot 10^{12} q_\Delta^3}\right\}. \tag{21}$$

For $V < 40/\varepsilon$, the latter bound is at least one, so the estimate is trivial. A union bound over seven coordinates per node costs $14n$ times the final exponential. There is no enumeration of count states or graph sizes.

5.3 Pathwise corridors and productive stock

On the complement of this one coordinate-noise event, differentiate only the continuous compensated primitive. Symmetric graph diffusion is controlled by the energy of positive barrier violations: its contribution is a nonpositive sum of edgewise squared differences. This gives comparisons on arbitrary finite weighted graphs.

The material comparison and strong induction over possible exit jumps remove the stop. For every node and $0 \le t \le 4$,

$$|A_i(t) - 1|, |B_i(t) - 1| \le 0.04e^{-t} + 0.0018, \tag{22}$$

$$h_i(t) < 0.009 + 0.00201e^{-3.02t} + 0.0003. \tag{23}$$

The assumptions at the post-pulse start are material deviations at most 0.04, $h_i(0) \le 0.01101$ and $Y_i(0) \ge 0.012$. The graph noise in the intermediate filter is bounded by $(2\Delta + 126/25)\varepsilon$, and

$$\varepsilon\left(1 + \frac{126/25 + 2\Delta}{151/50}\right) < 0.0003. \tag{24}$$

Equations (22)–(23) imply free-material amounts at least $0.94689 > 0.9$. At time four the material deviation is at most $0.0026 < 1/160$, and $h_i(4) < 0.00931 < 0.01101$. To see the first-exit argument explicitly, material noise is a weighted sum of coordinates with weight sum at most nine. Variation of constants and integration by parts bound its contribution by $9\varepsilon[1 + (1 + 2\Delta)\int_0^t e^{-s}\,ds] \le 18(1+\Delta)\varepsilon$. Thus the material deviation at the putative exit is at most $0.0418 < 0.1$, contradicting exit. For the intermediate filter, the damping is $1 + 101d_i \ge 3.02$ and its forcing is at most $0.009(1 + 101d_i)$: indeed $1.01d_i(1.1 + 121\eta_0) \le 0.009(1 + 101d_i)$ for $d_i \le 0.04$. The initial excess above 0.009 is at most 0.00201; convolution of the coordinate-noise error gives the additional 0.0003 allowance.

Let F_Y be the stock primitive. Its error from Y is at most $\eta = (11/2)\varepsilon \le 0.00055$. Use the barrier

$$F_Y(t) \ge \min\{0.0118e^{0.55t}, 0.052\}. \tag{25}$$

Indeed, the fence's initial value is below the available 0.012, its cap plus noise stays below the growth region's upper threshold 0.06, and the exact inequality

$$(2\Delta + 2/3)\eta \le (2/3 - 11/20)(59/5000) \tag{26}$$

pays for graph noise and the barrier growth rate. Positive-part comparison proves (25). A five-term exponential lower bound shows the cap is reached by $t = 11/4$. Therefore

$$Y_i(t) \ge 0.05145 \quad (11/4 \le t \le 4). \tag{27}$$

5.4 Phase transfer, marks and return

Write the productive phase vector as (x, c_1, c_2, z) and put

$$H = \begin{pmatrix} 0 & 20 & 0 & 38 \\ 0 & 5 & 20 & 0 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 20 & 24 \end{pmatrix}, \qquad M = H - 48\,\mathrm{Id}\,. \tag{28}$$

The literal compensated phase drift dominates M applied to the observed phase vector, plus graph diffusion. In the x row the count correction is favorable. In the z row we discard the entire nonnegative propensity $rN_X(N_X - 1)/V^2$, rather than discarding its negative correction while retaining a fictitious rx^2 term. For $\sigma = 1/28$, the row-vector adjoint

$$b(s) = e^{-48(\sigma - s)} \sum_{j=0}^{4} \frac{(\sigma - s)^j}{j!}\, e_1^\top H^j \tag{29}$$

is nonnegative, ends at $e_1^\top$, and starts above $\frac{1}{8}(1, 9/8, 7/5, 9/5)$. The omitted Taylor residual has the favorable sign. Differentiating $b(s)$ against the compensated primitive and applying graph comparison proves the uniform phase estimate

$$x_i(t) \geq \frac{0.05145}{8} - 8(1 + \Delta)\varepsilon = \frac{901}{160000} = 0.00563125 \qquad (3 \leq t \leq 4). \tag{30}$$

Only finite polynomial identities and exponential bounds are required; no assumption about a differentiable count path is made.

Integrability follows from the finite holding partition on every bounded time interval. Thus the actual compensators satisfy

$$\int_3^4 I_i \,\mathrm{d}t \geq 0.03675, \qquad \int_3^4 x_i \,\mathrm{d}t \geq 0.00563125, \qquad \int_0^4 g_i \,\mathrm{d}t \leq 0.18. \tag{31}$$

For the last bound, $g_i(t) \leq 0.0448924 < 0.045$ follows directly from $x_i \leq 1.1$, $h_i < 0.01131$ and $d_i \leq 0.04$.

The five physical marks have full-time compensation error less than $1/2000$ off an event of probability at most $10n \exp(-V/160000000)$. For this estimate the squared-mark bracket is at most $9/V$ and jumps are at most $2/V$; apply (35) at $a = 1/2000$ and union over five marks per node, since $4(9 + 2/2000)\,2000^2 < 160000000$. Taking differences at the two window endpoints costs less than $1/1000$. Hence normalized actual product counts are at least 0.03575 and 0.00463125, actual service is at most 0.181, and each continuous food count is at most 4.001. These product margins exceed $1/56$ and $1/1080$. Pulse refill is at most $0.755V$, and the actual marks are natural numbers. Consequently the ceil and floor inequalities in (11) hold exactly. The endpoint bounds above and (27) prove readiness.

For the illustrated flow, numerical quadrature gives inventory outputs 0.275164 and 0.275162, free-X outputs 0.128711 and 0.128710, and gross service 0.007416 and 0.007401 at nodes one and two. Each food input is 4.75, including refill. These normalized deterministic accounts illustrate the separate observables; the stochastic integer guarantees remain those of Theorem 3.

5.5 Pulse preparation and history induction

The categorical pulse moment bound, including floor refill and carried intermediate, gives preparation failure at most

$$4ne^{-V/100000} + ne^{-V/100000000}. \tag{32}$$

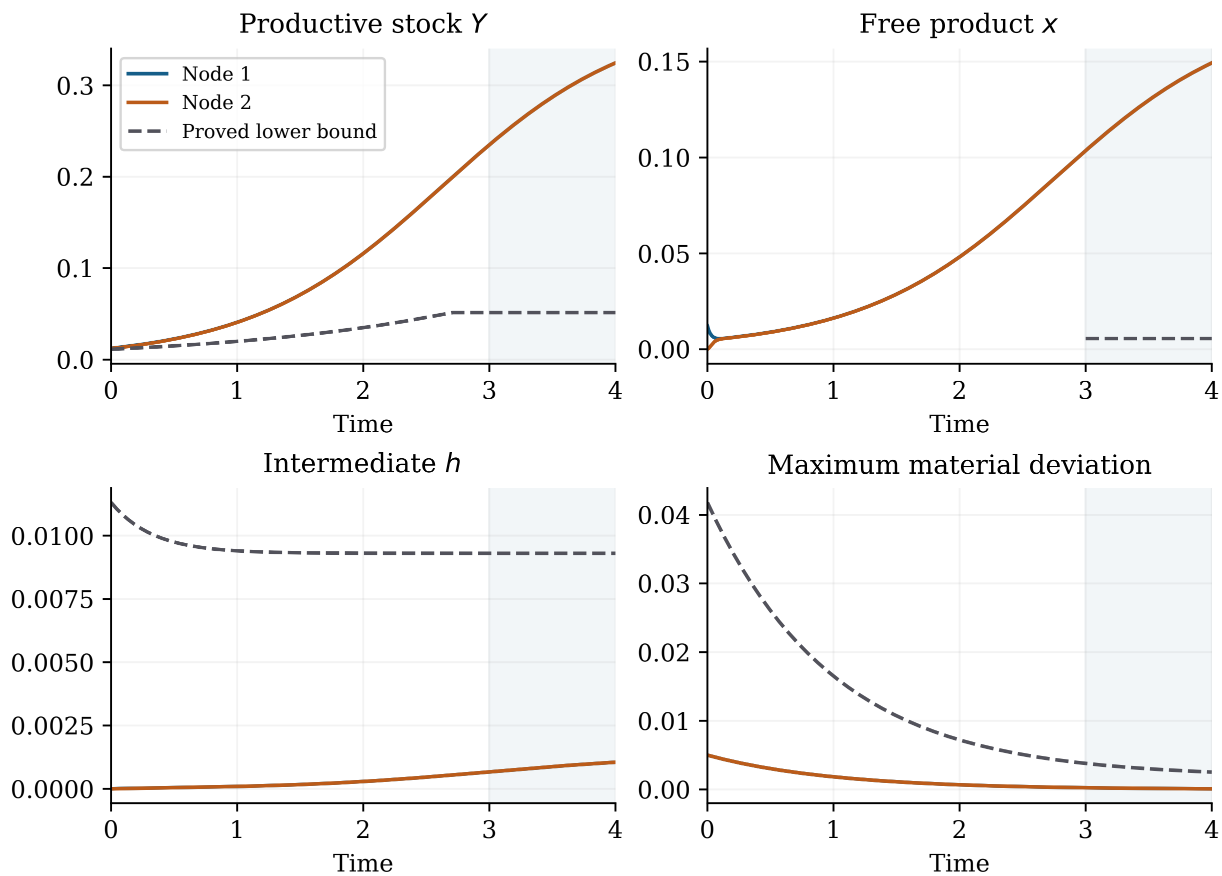

Figure 3: Illustrative deterministic flow for the two-node initial states (14), after $q = 1/4$, all $\ell_s = 0.98$, zero refill error, with $r_i = 20$, $d_i = 0.03$ and exchange rate one. Solid curves are numerical trajectories; dashed curves are the proved pathwise bounds (22)–(30). Shading marks the collection window. The numerical integration is not a stochastic simulation and supplies no proof evidence. Data and the short integration script accompany the paper.

For materials the squared-weight sum is at most $(161/80)V$, and for stock it is at most $(1449/800)V$. The conditional Bernoulli bound proved in Appendix B, at deviations $V/100$ and $V/4000$, implies these weakened denominators. The intermediate cannot increase at a pulse, so its preparation ceiling requires no additional tail. Floor refill costs less than $1/V$, leaving material error below 0.04 and stock at least 0.012. Each of these pulse and mark exponents dominates the final exponent in (10). Pulse preparation therefore costs $5n$, and operating noise costs $14n + 10n = 24n$, times that exponential. Integration over every pulse outcome gives the uniform one-cycle failure bound $29ne^{-V/(2\cdot 10^{12}(1+\Delta)^3)}$.

Let the history invariant mean that every logged output satisfies (11) and the current state is ready. From an invariant history the physical conditional law preserves this invariant with probability at least one minus the one-cycle error. Induction through the unrestricted Markov history kernel loses at most that error per cycle. Nonnegativity supplies the outer maximum with zero. This proves Theorem 3 without conditioning the law on successful histories.

More explicitly, if E_j denotes success of the first j cycles and $p = 29n\exp[-V/(2\cdot 10^{12}(1+\Delta)^3)] \leq 1$, then

$$\mathbb{P}(E_{j+1}) = \mathbb{E}\Big[\mathbf{1}_{E_j}\mathbb{P}(\text{next success} \mid \mathcal{F}_j)\Big] \geq (1-p)\mathbb{P}(E_j).$$

Consequently $(1-p)^m \geq 1-mp$. For $p>1$ and $m \geq 1$ the stated bound is zero; $m=0$ is immediate. This induction does not assert independent cycles.

6 Sustained synthesis and resource interpretation

Proposition 8 (Augmented-inventory accounting)**.** *Let $J(N) = N_X + N_{C_1} + N_{C_2} + 2N_Z + N_D$. For each chemical pair j in Table 1, let H_j be its net forward event count over the whole mission and all nodes, starting just before the first pulse and ending at the last return. Set $S_J = H_0 + H_3 - H_6$. Every finite physical path satisfies*

$$S_J = J_{\text{final}} - J_{\text{initial}} + J_{\text{washout}} + J_{\text{withdrawn}} + J_{\text{pulse loss}}. \tag{33}$$

On a successful m-cycle mission with $m \geq 1$,

$$\begin{aligned} S_J &\geq nm\lceil V/56 \rceil + nV/28 - J_{\text{initial}} \\ &\geq nV\left(\frac{m}{56} + \frac{1}{28} - \frac{161}{160}\right). \end{aligned} \tag{34}$$

The last expression is positive for $m \geq 55$ and equals $(913/1120)nV$ for $m = 100$. These consequences hold with the same mission probability lower bound.

Proof. The J increments of the seven forward pairs are $(1,0,0,1,0,0,-1)$; reverse increments are their negatives. Feed and refill have zero J weight. Every exchange event cancels globally, and each pulse category removes its actual inventory weight. Summing these increments proves (33); nonexplosion ensures finite event counts. On success, all washout includes the credited inventory collection, so $J_{\text{washout}} \geq nm\lceil V/56\rceil$. Coefficientwise, $J \geq I \geq 5Y/7$, hence $J_{\text{final}} \geq nV/28$. Initially $J \leq A \leq 161V/160$ at each node. Discarding nonnegative pulse losses proves (34). Finally the bracket equals $(20m - 1087)/1120$, which is positive precisely for integer $m \geq 55$. This is an eventwise consequence, requiring no further probability cost. □

The intermediate is included in the carried inventory but not credited as product. Thus a long successful mission cannot be explained by draining its initial intermediate and productive stock. S_J measures net chemical creation of inventory equivalents; it is not a thermodynamic efficiency or a claim about the identity of atoms in an experiment. Also Q_X is part of Q_I; these outputs must not be added as disjoint products.

In addition to (13), success gives the division-free service inequalities

$$G \leq \frac{56}{5}Q_I, \qquad G \leq 216Q_X.$$

They follow from $G \leq V/5$, $Q_I \geq V/56$ and $Q_X \geq V/1080$ and sum over the mission. At fixed n, V, Δ and failure allowance $0 < \delta < 1$, a sufficient mission duration is

$$m \leq \left\lfloor \frac{\delta}{29n} \exp\left(\frac{V}{2 \cdot 10^{12}(1+\Delta)^3}\right)\right\rfloor,$$

provided the readiness and $V \geq 10000$ conditions hold. A zero right side means that the certificate supplies no positive mission at this allowance, not certain failure.

7 A worked operating mission

For the connected example (14), $n = 2$, $m = 100$, $\Delta = 1$ and $V = 2.24 \cdot 10^{14}$. The certified success lower bound is $1 - 5800e^{-14} \simeq 0.9951771334$. On this event the guaranteed total credited inventory is at least $8 \cdot 10^{14}$ equivalents, while (34) guarantees $S_J \geq 3.652 \cdot 10^{14}$. The initialization is literal and integral since V is divisible by 20 and 28. The probability guarantee is uniform over the stated controller class.

Figure 3 shows the first deterministic cycle with fixed $r_i = 20$, $d_i = 0.03$, $q_i = 1/4$, $\ell_{is} = 0.98$, zero refill error and exchange rate one. Figure 4 repeats this same pulse from the actual computed seven-species endpoint for ten cycles, including intermediate carryover. These coupled deterministic trajectories illustrate the model; the local deterministic theorem does not by itself certify a deterministic graph extension. Integration uses Radau with relative tolerance $2 \cdot 10^{-9}$

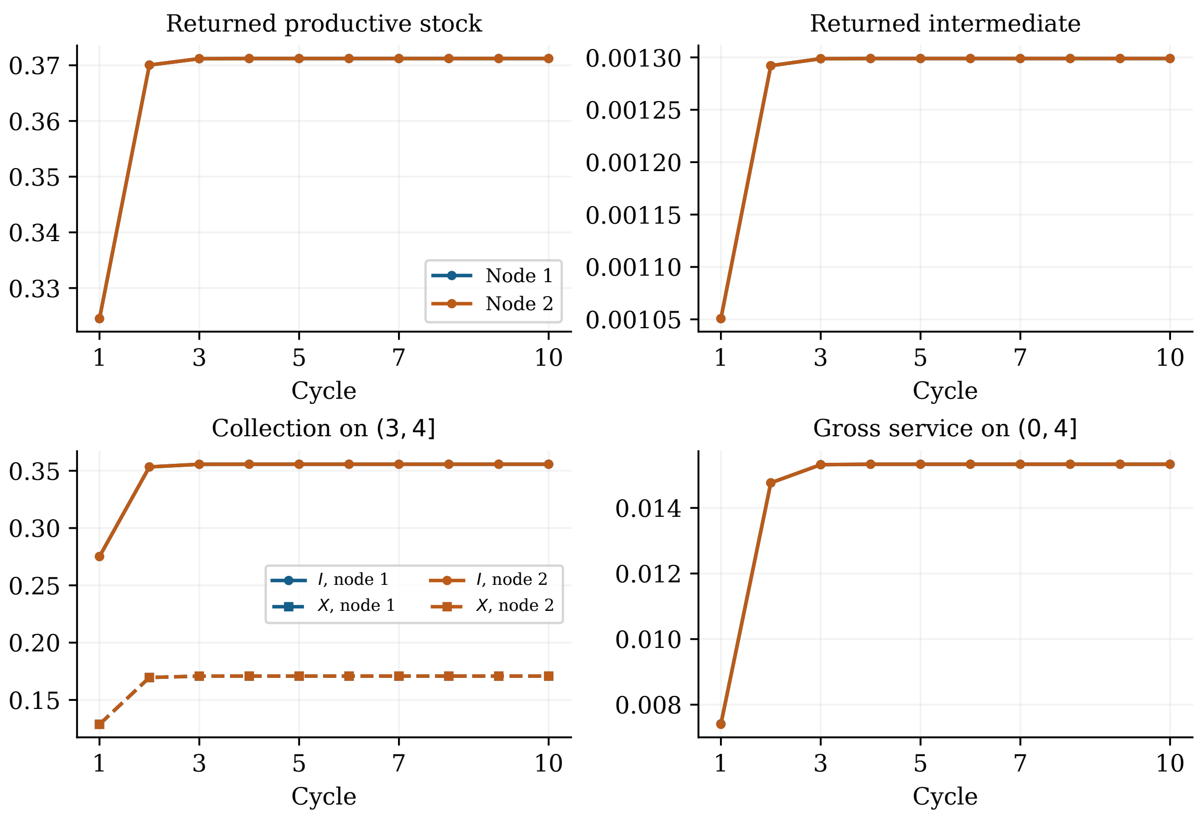

Figure 4: Ten deterministic cycles with actual endpoint carryover. Upper panels show returned stock and intermediate. Lower panels separate collected inventory, collected free X and gross service (normalized by count scale); each food account is 4.75 per cycle. Solid curves are numerical results, not a sample of the molecular law. The accompanying CSV records every node and cycle.

and absolute tolerance $2 \cdot 10^{-11}$. Product and service integrals are accumulated as additional ODE coordinates, splitting the flow at the collection boundary. A rerun with both tolerances reduced by a factor of 100 changes the recorded endpoint and account values by less than $6 \cdot 10^{-13}$; neither simulation nor quadrature is proof evidence.

To attach units, choose concentration unit c_* ($\mathrm{mol\,L^{-1}}$) and time unit τ (seconds). Then $c_{\text{physical}} = c_* c$, $t_{\text{physical}} = \tau t$ and

$$V = N_A \Omega c_*, \qquad N_A = 6.02214076 \cdot 10^{23}\ \mathrm{mol}^{-1},$$

where Ω is a reactor's volume in litres. A cycle lasts 4τ; a rate-one washout corresponds to $1/\tau$; each unit feed corresponds to c_*/τ. A dimensionless second-order coefficient k corresponds to $k/(c_*\tau)$. At $c_* = 1\,\text{mM}$ the witness scale corresponds to $\Omega \simeq 0.372\,\mu\text{L}$; at $c_* = 1\,\mu\text{M}$ it corresponds to $\Omega \simeq 0.372\,\text{mL}$. This conversion is illustrative: no particular c_*, τ, chemical identity, experimental rates or membrane have been calibrated.

8 Discussion and verification scope

The common event is operational: counted effluent, bounded input and transaction counts, and recovery of the actual state. Material histories make the donor reduction exact; intermediate storage explains why a naive projection of the refined source fails. Useful operation can nevertheless be proved without a uniform approximation of all species trajectories. The synthesis telescope strengthens this interpretation by accounting for all initially carried intermediate.

For bounded weighted degree, the sufficient per-node count scale grows logarithmically with node count, mission length and inverse failure allowance. The cubic degree factor and large numerical constants are features of this sufficient proof. They are not lower bounds, optimal scaling laws, or a monotonic prediction of how transport changes physical reliability. The result begins in readiness and concerns finite missions; it does not assert food-only startup, all-time persistence, arbitrary intermediate lifetimes, selective or asymmetric transport, asynchronous pulses, or universal inheritance under reaction refinement.

The refined deterministic theorem, the finite-count mission theorem, its concrete returned-history construction, the connected witness, and the food-to-output bounds are supported by Lean proofs. The recorded strict publication verification succeeds with warnings treated as errors, and the checked declarations report only `propext`, `Classical.choice` and `Quot.sound`. The source and dependency hashes were compared with the saved receipt for this paper. The six-species probability theorem is conventional; selected donor algebra and margins have separate Lean checks. Propositions 6, 7 and 8, and the service/duration corollaries, are supplied here as conventional proofs. We do not describe the entire manuscript as formally verified.

The editable source distribution includes the bibliography, vector figures, numerical data, reproduction scripts and a claim-to-source map. Formal proof source paths and the frozen verification environment are identified in that map. The numerical figures are illustrations of the literal equations and not estimates of stochastic failure rates.

A Quantitative source estimates

For completeness, the estimates behind the stock and phase steps can be checked directly from the displayed chemistry. Put $A_0 = A - h$, $B_0 = B - h$. If $A_0, B_0 \geq 0.9$ and $Y \leq 0.06$, then $u \geq 119/150$, $w \geq 57/70$, $x^2 \leq (3/50)x$. Expansion of the refined drift gives

$$Y(f) = \epsilon_0 uw + (\tfrac{5}{2}u - 1 - d(1+\beta) - \epsilon_0/10)x + (\tfrac{11}{2}w - \tfrac{29}{8})c_1 \\ + \tfrac{11}{10}c_2 + (r/5 - 13/5)z - (r/5)x^2 + d\beta h/\theta.$$

Subtracting $2Y/3$ leaves nonnegative coefficients: the x surplus is at least $37/1500 - d\beta - \epsilon_0/10 > 0$; those of c_1, c_2, z are at least $29/280$, $1/6$ and $(r-19)/5$. The count correction contributes $rx/(5V) \geq 0$. Also $I - 5Y/7 = (2/7)x + (11/56)c_1 + (5/7)z \geq 0$.

For the matrix H in (28), the first row of its degree-four exponential polynomial at $1/28$ is

$$\left(1, \frac{961355}{1229312}, \frac{1847785}{1843968}, \frac{665611}{307328}\right).$$

Dividing by the stock weights gives minimum ratio 961355/1382976. The exponential series with a geometric remainder gives $e^{6/7} \le \sum_{j=0}^{7}(6/7)^j/j! + (6/7)^8(9/8)/8! \le 2357/1000$, whose square is at most 961355/172872. Hence multiplying the row by $e^{-12/7}$ dominates the stock weights divided by eight. The x loss is bounded by $1 + d(1+\beta) + \epsilon_0/10 + 42(u+x) < 48$; the other diagonal losses are bounded by $43, 41, 24$. This proves the lower phase system including the literal duplex propensity.

An alternative conventional derivation of the phase-noise estimate makes its constant explicit. On a network put $K = \mathcal{D} \otimes \mathrm{Id}_4 + \mathrm{Id}_n \otimes M$. Then $\|e^{Ks}\|_\infty \le e^{10s}$ and $\|K\|_\infty \le 106 + 2\Delta$. Integration by parts on a window of length 1/28 bounds the common coordinate-noise contribution by

$$\varepsilon \left[1 + e^{10/28} + (106 + 2\Delta) \int_0^{1/28} e^{10s}\, ds\right] < 8(1+\Delta)\varepsilon.$$

Here $e^{5/14} < 3/2$ and the integral is below 1/20. The positive graph semigroup averages the uniform starting stock floor. This yields (30); the formal proof implements the finite adjoint.

For the deterministic theorem the material deviation immediately after the pulse is below 0.04 and subsequently decays as $0.04e^{-t}$. The filter ceiling $h \le 1.101\theta$ is invariant: under materials at most 1.1 its forcing $x + (\eta_0/\beta)uw < 1.101$, so the derivative is negative at the ceiling. Thus $A_0, B_0 > 0.9$. Starting at $Y \ge 49/4000$, the preceding growth inequality gives $Y(t) \ge \min\{(49/4000)e^{2t/3}, 0.06\}$ and hence $Y \ge 1/20$ by $t = 5/2$. Material decay and the filter ensure readiness at time four. The same positive phase system without noise gives at least $Y/8$ in free x after 1/28 time, which is stronger than the asserted 1/540 collection floor. Finally $g \le 112201/2500000$, and each food input is $5 - q + e \le 951/200$. Integration and recursion prove Theorem 2 at every fixed admissible θ.

B Conventional donor probability proof

The donor uses the same nonexplosion argument and phase matrix. Its stock expansion is the one above with $-d(1+\beta)x + d\beta h/\theta$ replaced by $-dx + d\eta_0 uw$. The surplus is therefore at least the displayed donor coefficients. Here are explicit estimates closing its probability proof, without a six-species instantiation of the refined formal probability root.

For a compensated jump martingale with jumps at most b/V in absolute value and bracket at time T at most v/V,

$$\mathbb{P}(\sup_{t\le T} |M_t| \ge a) \le 2\exp[-Va^2/(4(v+ba))]. \tag{35}$$

Indeed $e^u - 1 - u \le u^2$ for $|u| \le 1$ makes $\exp(\lambda M - \lambda^2\langle M\rangle)$ a nonnegative local supermartingale when $\lambda b/V \le 1$. Localize, apply the crossing bound and Fatou's lemma, choose $\lambda = Va/[2(v+ba)]$, and repeat for $-M$. The bound includes the exit jump because brackets use its pre-jump intensity.

Conditional pulse means lie in $[31213/32000, 3231/3200]$ for both materials before floor rounding, and stock mean is at least 49/4000. The sums of squared weights are at most $(161/80)V$ and $(1449/800)V$ for material and stock. The centered Bernoulli log-moment bound $\log \mathbb{E}e^{\lambda a(B-\mathbb{E}B)} \le \lambda^2 a^2/8$ follows by integrating the tilted variance bound 1/4 twice. Applying it at deviations $V/100$ and $V/4000$ gives pulse failure at most $4ne^{-V/11000} + ne^{-V/15000000}$. On its complement the post-pulse material error is at most 0.04 and stock at least 0.012.

Stop at first material exit from $[0.9, 1.1]$, including its jump. The sum of local propensities divided by V is below 200, by direct substitution of the corridor and rate maxima in the six chemical pairs, feeds and washouts. Coordinate jumps have size at most $2/V$; exchange contributes at most $2.2\Delta/V$ to each coordinate squared-jump rate. Thus the four-time bracket is at most $3200(1+\Delta)/V$. Set $\varepsilon_0^* = 1/[100000(1+\Delta)]$. Equation (35) and a union over $6n$ coordinates cost $12n\exp[-V/(2\cdot 10^{14}(1+\Delta)^3)]$.

On this common event, material martingale weights sum to at most eight. Writing $L = \mathcal{D} - \mathrm{Id}$, integration by parts gives

$$A_0(t) - \mathbf{1} = e^{Lt}(A_0(0) - \mathbf{1}) + M_A(t) + \int_0^t L e^{L(t-s)} M_A(s)\, ds.$$

Since $\|e^{Lt}\|_\infty \le e^{-t}$ and $\|L\|_\infty \le 1 + 2\Delta$, the material deviation is at most $0.04e^{-t} + 0.00016$. At any first exit this is strictly below 0.1, a contradiction, so the stop is removed. The compensated stock is above $\min\{0.0119e^{0.6t}, 0.059\}$: on a sufficiently narrow strip below this fence, its derivative minus the fence derivative is positive because $(2/3 + 2\Delta)(11/2)\varepsilon_0^* \le 0.00011$ and the growth surplus is at least $0.0119/15$. Integrating on a hypothetical downward crossing of the strip proves comparison. The cap is reached by $11/4$, yielding $Y > 0.058$. The phase estimate above then gives $x > 0.007$ on $[3, 4]$ and $I > 0.04$. Endpoint material error is below $0.001 < 1/160$, so readiness returns.

The five stopped marks have jumps at most $2/V$ and brackets at most $9/V$. Using (35) at $a = 0.001$, with collection marks gated to $(3, 4]$, costs $10ne^{-V/40000000}$. Actual normalized product is then at least 0.039 for Q_I and 0.006 for Q_X. The service compensator is at most 0.18, since $dx + d\eta_0 uw < 0.045$ on the corridor; actual service is at most 0.181. Each food account is at most $0.755 + 4 + 0.001 = 4.756$. These strict inequalities imply all integer ceilings and floors for $V \ge 10000$. The prefactors $5 + 12 + 10 = 27$ give the common one-cycle bound after weakening exponents. Conditional induction from the actual returned state, as in the main proof, proves Theorem 4.

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